Review : Elements of Fourier Series

① Periodic function

- A periodic function is a function which satisfies the relation

$$f(x) = f(x+T) \quad \text{for all } x \text{, and a given } T>0$$

T is the <u>period</u> of the function.

- Note that a function which is periodic with period T is <u>also</u> periodic with period nT for any $n \in \mathbb{N}$, $n>0$. Usually T is the smallest real value for which $f(x) = f(x+T)$ holds.

② Orthogonality

- An inner product can be defined for any two functions on an interval $[a,b]$ as

$$\langle f,g \rangle = \int_a^b f(x)g(x)w(x)\,dx$$

where $w(x)$ is a fixed positive weight function (usually satisfying $\int_a^b w(x)dx = 1$.)

- Two functions are therefore orthogonal on $[a,b]$ provided $\langle f,g \rangle = 0$.

Property: with $w(x) = \frac{1}{2L}$

- the functions $\sin\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{m\pi x}{L}\right)$ are orthogonal on $[-L,L]$ for all (m,n)
- the functions $\sin\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{m\pi x}{L}\right)$ are orthogonal on $[-L,L]$ for all $m \neq n$
- the functions $\cos\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{m\pi x}{L}\right)$ are orthogonal on $[-L,L]$ for all $m \neq n$

③ Fourier Series

Any function f periodic with period $2L$ can be written as the series (called a Fourier Series)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

the Fourier coefficients

$$\begin{cases} a_0 = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx \\[2ex] a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\[2ex] b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \end{cases}$$

<u>Proof</u>: Let $m > 0$

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx$$

$$= \int_{-L}^{L} \left[a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right] \cancel{\cos}\left(\frac{m\pi x}{L}\right) dx$$

$$= \int_{-L}^{L} a_0 \cos\left(\frac{m\pi x}{L}\right)dx + \int_{-L}^{L}\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

$$+ \int_{-L}^{L}\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

given some prayer for convergence of the series

$$= \int_{-L}^{L} a_m \cos^2\left(\frac{m\pi x}{L}\right)dx = \frac{2L}{2} a_m = L a_m$$

so $$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx$$ as required

(and similarly for the other terms)

④ Properties of Fourier a_n, b_n coefficients

- if $f(x)$ is an even function $(f(x) = f(-x))$ then $b_n = 0 \quad \forall n \implies$ keep only the constant & cosine terms
- if $f(x)$ is an odd function $(f(x) = -f(-x))$ then $a_n = 0 \quad \forall n \implies$ keep only the sin terms.
- The Fourier coeffs. of the sum of two functions f and g is equal to the sum of the Fourier coefficients

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$g(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)$$

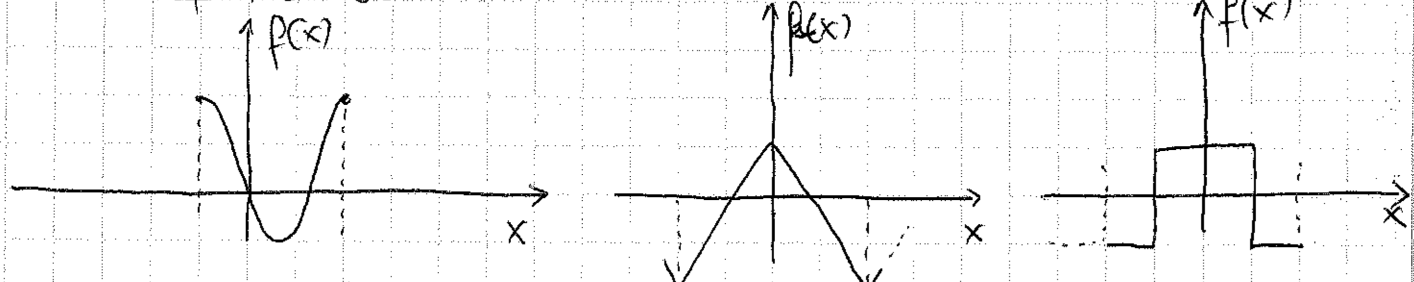
then

$$f+g\ (x) = (a_0 + A_0) + \sum_{n=1}^{\infty} (a_n + A_n) \cos\left(\frac{n\pi x}{L}\right) + (b_n + B_n) \sin\left(\frac{n\pi x}{L}\right)$$

<u>BUT</u> : not true for the product!

- The Fourier series can be differentiated/integrated term by term to obtain the Fourier series of the derivative/integral of a function. (see Textbook 3.4 & 3.5)

- The smoother the function, the quicker the convergence of the series:



$a_n, b_n \sim \frac{1}{n^3}$ or faster $\qquad a_n, b_n \sim \frac{1}{n^2} \qquad a_n, b_n \sim \frac{1}{n}$.

→ This is a good way of checking your answer.

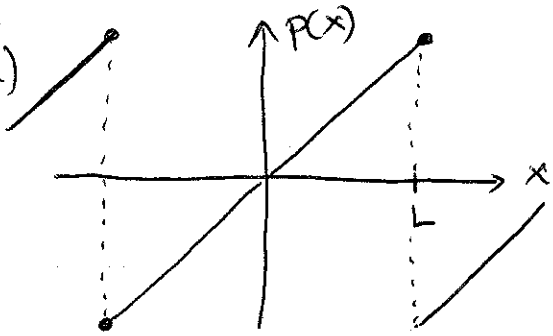
(5) Examples

Example 1: Consider the function $f(x) = x$ on $[0, L]$.

(a) • What is the Fourier series of the function $p(x)$ which is periodic with period L, and odd?

(b) • " " " " .... and even?

(a)



← The odd & periodic extension of f looks like this.

$$p(x) = \cancel{a_0} + \sum_{n=1}^{\infty} \overbrace{\cancel{a_n \cos\left(\frac{n\pi x}{L}\right)}}^{0 \text{ by symmetry}} + b_n \sin\left(\frac{n\pi x}{L}\right)$$

with

$$b_n = \frac{1}{L}\int_{-L}^{L} p(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{2}{L}\left\{\left[-x\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L + \int_0^L \frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)dx\right\}$$

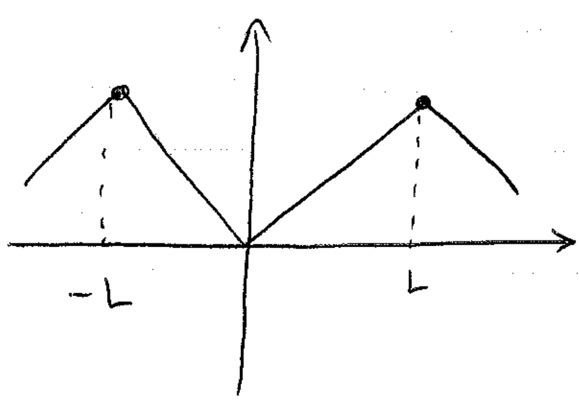
$$= \frac{2}{L}\left\{-\frac{L^2}{n\pi}\cos(n\pi) + \left[\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\right]_0^L\right\}$$

$$= -\frac{2L}{n\pi}\cos(n\pi).$$

So

$$p(x) = -\sum_{n=1}^{\infty}\frac{2L}{n\pi}\cos(n\pi)\sin\left(\frac{n\pi x}{L}\right)$$

(b)



← the even & periodic extension of $f(x)$ looks like this

$$p(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)$$

$$+ \overbrace{\cancel{b_n \sin\left(\frac{n\pi x}{L}\right)}}^{0}$$

with $\quad a_0 = \frac{1}{2L}\int_{-L}^{L} p(x)\,dx = \frac{1}{L}\int_0^L x\,dx = \frac{1}{L}\cdot\frac{L^2}{2} = \frac{L}{2}.$

$$a_n = \frac{1}{L}\int_{-L}^{L} p(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\left\{\left[x\cdot\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^L - \int_0^L \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)dx\right]$$

$$= \frac{2}{L}\left\{ 0 + \left[\frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L\right\}$$

$$= \frac{2L}{n^2\pi^2}\left(\cos(n\pi) - 1\right)$$

So

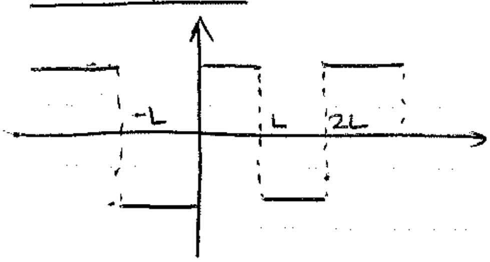
$$p(x) = \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L}{n^2\pi^2}\left(\cos(n\pi)-1\right)\cos\left(\frac{n\pi x}{L}\right).$$

Comparing the two cases, we see that

(a) has $p(x)$ non-continuous. Note how the coefficients $b_n \propto \frac{1}{n}$. Also note that at $x = \pm L$, $p = 0$ (even though it is not in the original function).

(b) has $p(x)$ continuous but its derivative is not. We see $a_n \propto \frac{1}{n^2}$.

<u>Example 2</u> : Consider the periodic function $g(x)$ drawn here.



The Fourier Series of $g(x)$ is (by symmetry)

$$g(x) = \sum_n b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{with}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} g(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)$$

$$= -\frac{2}{L}\cdot\frac{L}{n\pi}\left(\cos(n\pi)-1\right) = -\frac{2}{n\pi}\left(\cos(n\pi)-1\right)$$

Note first that, as expected, the convergence is $\propto \frac{1}{n}$.

Also, $g(x)$ is in fact the derivative of $p(x)$ determined in Example 1,b. It's easy to verify that a term-by-term differentiation of $p(x)$ leads to the same answer:

$$g(x) = p'(x) = \left[\frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L}{n^2\pi^2}\left[\cos(n\pi)-1\right] \cos\left(\frac{n\pi x}{L}\right)\right]'$$

$$= \sum_{n=1}^{\infty} \frac{2\cancel{L}}{n^{\cancel{2}}\pi^{\cancel{2}}}\left(\cos(n\pi)-1\right)\left(-\frac{\cancel{n}\cancel{\pi}}{\cancel{L}}\right)\cdot \sin\left(\frac{n\pi x}{L}\right)$$

$$= -\sum_{n=1}^{\infty} \frac{2}{n\pi}\left(\cos(n\pi)-1\right) \sin\left(\frac{n\pi x}{L}\right)$$

as required ✓

(H) Back to the original question:
How to impose arbitrary initial conditions to our diffusion problem?

<u>Recap</u>:
$$\begin{cases} \dfrac{\partial u}{\partial t} = k \dfrac{\partial^2 u}{\partial x^2} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$f(x)$ arbitrary but <u>known</u> in $[0,L]$.

We were able to express the general solution to the PDE + BCs as

$$u(x,t) = \sum_{n=1}^{\infty} \alpha_n e^{-\frac{n^2\pi^2}{L^2}kt} \sin\left(\frac{n\pi x}{L}\right).$$

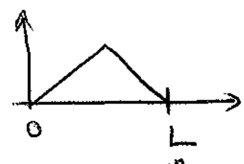
To apply the IC we want $f(x) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{L}\right)$

$\Rightarrow$ This <u>looks</u> like a Fourier Series for a function that is

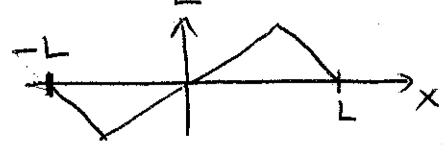
- periodic with period $2L$
- odd in x.

Since $f(x)$ is only defined in $[0,L]$, we <u>extend it</u> on $[-L,L]$ and make sure the extension $\bar{f}(x)$ is odd

<u>Example</u>: if $f(x) =$



then $\bar{f}(x) =$



On $[-L,L]$ we now have $\bar{f}(x) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi x}{L}\right)$

so
$$\boxed{\alpha_n = \frac{1}{L}\int_{-L}^{L} \bar{f}(x) \sin\left(\frac{n\pi x}{L}\right) dx}.$$

$$\alpha_n = \frac{1}{L}\int_{-L}^{L} \bar{f}(x)\sin\left(\frac{n\pi x}{L}\right)dx$$ can then be rewritten as

$$\boxed{\alpha_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx} \quad (+)$$

This last expression only depends now on the original $f(x)$ & can be used to find α_n.

With time & experience, you will become more comfortable going straight to $(+)$. However, I find that this manner of thinking about the problem makes things easier, in particular when working with other kinds of BCs, basic functions, etc... Also, it makes it easier to get the factors of 2 right.

<u>Example</u>: if
$$\begin{cases} f(x) = x & \text{in } [0, \frac{L}{2}] \\ f(x) = L - x & \text{in } [\frac{L}{2}, L] \end{cases}$$

then

$$\alpha_n = \frac{2}{L}\int_0^{L/2} dx\, x \sin\left(\frac{n\pi x}{L}\right) + \frac{2}{L}\int_{L/2}^{L} dx (L-x)\sin\left(\frac{n\pi x}{L}\right)$$

$$= \frac{2}{L}\left\{\frac{L^2}{\pi^2 n^2}\left(\sin\left(\frac{n\pi}{2}\right) - \cancel{\frac{n\pi}{2}\cos\left(\frac{n\pi}{2}\right)}\right)\right.$$

$$\left. + \frac{L^2}{\pi^2 n^2}\left(\sin\left(\frac{n\pi}{2}\right) + \cancel{\frac{\pi n}{2}\cos\left(\frac{n\pi}{2}\right)}\right)\right\}$$

$$= \frac{4L}{\pi^2 n^2}\sin\left(\frac{n\pi}{2}\right).$$

$\Rightarrow$ finally

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4L}{\pi^2 n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\pi^2}{L^2}kt}.$$

(see movie) ...... n=1 mode - - - n=2 mode —·—·

The movie illustrates how

- the higher-order Fourier modes dissipate faster. After some time, most of the "fine-structure has disappeared, and only the largest-scale mode is left.

- The evolution of the large-scale mode follows

$$u \propto e^{-\frac{\pi^2}{L^2} k t}$$

$\rightarrow$ diffusion timescale, overall, is $\boxed{\tau = \frac{L^2}{k\pi^2}}$.

NOTE This could actually have been deduced, without calculations, from the original equation:

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

$$\Rightarrow \quad \frac{u}{\underbrace{\tau}_{\text{characteristic timescale}}} = \frac{k u}{\underbrace{L^2}_{\text{characteristic lengthscale}}} \quad \leftarrow \text{roughly speaking}$$

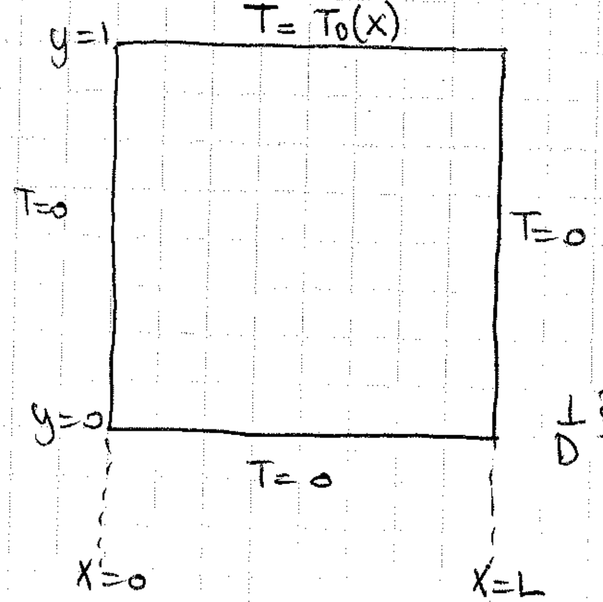
Which implies $\tau = \frac{L^2}{k}$.

This is not too different from the one obtained much more rigorously above.

This "dimensional analysis" is a very general type of approach used to determine "order of magnitude" estimates of the behavior of a system. We will see more of this later...

4.3 Laplace Equation

Consider a square plate with sides held at the following temperatures:



What is the steady-state temperature profile in the plate as a result of this heating?

→ Solve

$$\frac{1}{D}\frac{\partial T}{\partial t} = \boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0}$$

Separation of variables:

$$T(x,y) = A(x)\, B(y)$$

$$\Rightarrow \quad \frac{1}{A}\frac{d^2A}{dx^2} = K$$

$$\frac{1}{B}\frac{d^2B}{dy^2} = -K$$

Note that
- if $K>0$ then $\begin{cases} A \text{ has exponential behaviour} \\ B \text{ has oscillatory behaviour} \end{cases}$
- if $K=0$ then both must be linear
- if $K<0$ then $\begin{cases} A \text{ has oscillatory behaviour} \\ B \text{ has exponential behaviour.} \end{cases}$

- looking at the boundary conditions in x ($A(0) = A(L) = 0$) we see that if A is a linear combination of $e^{\sqrt{K}x}$ and $e^{-\sqrt{K}x}$ then the only solution is $A=0$

$$\rightarrow \quad K \leq 0$$

- We can rule out $K=0$ on the same ground

$\rightarrow$ $K<0$ so define $K = -k^2$

$\Rightarrow$ for each k,

$$A_k(x) = a_k \cos kx + b_k \sin kx$$

$$B_k(y) = \alpha_k e^{ky} + \beta_k e^{-ky}$$

or equivalently

$$= \tilde{\alpha}_k \cosh(ky) + \tilde{\beta}_k \sinh(ky)$$

$$A_k(0) = A_k(L) = 0 \quad \Rightarrow \quad a_k = 0 \qquad k = \frac{n\pi}{L}$$

$$B_k(0) = 0 \quad \Rightarrow \quad \tilde{\alpha}_k = 0$$

So $\quad T(x,y) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right)$

(note that the term for $n=0$ is 0).

To satisfy the remaining boundary condition at $y=1$ we require that

$$T(x,1) = T_0(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi}{L}\right)$$

$\Rightarrow$ This looks like a Fourier Series for an odd function periodic with period $2L$ $\Rightarrow$ let's construct the $\tilde{T}_0(x)$ extension of $T_0(x)$ with these properties, then

$$\sinh\left(\frac{n\pi}{L}\right) b_n = \frac{1}{L}\int_{-L}^{L} \tilde{T}_0(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{\sinh\left(\frac{n\pi}{L}\right)} \frac{2}{L} \int_0^L T_0(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

<u>Example</u>: Let $\quad T_0(x) = A\sin^2\left(\frac{\pi x}{L}\right)$ then

$$\frac{2}{L}\int_0^L A\sin^2\left(\frac{\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{\cancel{2}}{L}\int_0^L A\left(\frac{1-\cos\left(\frac{2\pi x}{L}\right)}{\cancel{2}}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

Using $\cos a \sin b = \frac{1}{2}\sin(a+b) - \frac{1}{2}\sin(a-b)$

$$= \frac{A}{L}\left\{ \left[-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L - \frac{1}{2}\int_0^L \sin\left(\frac{(n+2)\pi x}{L}\right) dx + \frac{1}{2}\int_0^L \sin\left(\frac{(2-n)\pi x}{L}\right) dx \right\}$$

$$= \frac{A}{L}\left\{ \left(-\frac{L}{n\pi}\cos(n\pi) + \frac{L}{n\pi}\right) - \frac{1}{2}\frac{L}{(n+2)\pi}\left((-1)^{n+2} - 1\right) - \frac{1}{2}\frac{L}{(n-2)\pi}\left((-1)^{n-2} - 1\right)\right\}$$

if $n \neq 2$.

$\Rightarrow$ if n is even, $b_n = 0$.

if n is odd then

$$b_n = \frac{A}{L}\left\{ + \frac{2L}{n\pi} + \frac{L}{(n+2)\pi} + \frac{L}{(n-2)\pi} \right\} \cdot \frac{1}{\sinh\left(\frac{n\pi}{L}\right)}$$

$$= \frac{1}{\sinh\left(\frac{n\pi}{L}\right)} \left[\frac{2}{n\pi} + \frac{2n}{(4-n^2)\pi}\right] A = \frac{8A}{n(4-n^2)\pi} \frac{1}{\sinh\left(\frac{n\pi}{L}\right)}$$

so finally,

$$T(x,y) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8A}{n(4-n^2)\pi} \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right) \frac{1}{\sinh\left(\frac{n\pi}{L}\right)}$$

$$= \sum_{p=0}^{\infty} \frac{8A \cdot \frac{1}{\sinh\left(\frac{(2p+1)\pi}{L}\right)}}{(2p+1)(4-(2p+1)^2)} \sin\left(\frac{(2p+1)\pi x}{L}\right) \sinh\left(\frac{(2p+1)\pi y}{L}\right)$$

Note: we can see that if $A = 0$ $(T_0(x) = 0)$ then the solution in the domain is identically 0

$\Rightarrow$ This is a property of Laplace's equation: if the bcs are identically 0 on the contour then the solution is 0 everywhere. We will see more of this later in the course