CHAPTER 2 Method of separation of variables for linear equations with constant coefficients

In this chapter we explore a method for the solution of PDES which works, under certain circumstances, to solve linear 2nd (and higher) order eqs with constant coefficients, and with some additional work, linear equations with non-constant coefficients too.

The method is fundamentally based on the linearity of the PDE, and its consequence for the principle of superposition

① Linearity & the principle of superposition

Let us write the PDE as $\mathcal{L}(u) = f$
where $\mathcal{L}$ is the operator acting on the function u.
and f is known

eg For the heat equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

$$\begin{cases} \mathcal{L} = \frac{\partial}{\partial t} - k\frac{\partial^2}{\partial x^2} \\ f = 0 \end{cases}$$

Definition: an operator $\mathcal{L}$ is linear if

$$\mathcal{L}(c_1 u_1 + c_2 u_2) = c_1 \mathcal{L}(u_1) + c_2 \mathcal{L}(u_2)$$

where u_1 & u_2 are two functions and c_1 & c_2 two constants e.g. For the heat equation, we verify that

$$\mathcal{L}(c_1u_1+c_2u_2) = \frac{\partial}{\partial t}(c_1u_1+c_2u_2) - k\frac{\partial^2}{\partial x^2}(c_1u_1+c_2u_2)$$
$$= c_1\left(\frac{\partial u_1}{\partial t} + k\frac{\partial^2 u_1}{\partial x^2}\right) + c_2\left(\frac{\partial u_2}{\partial t} + k\frac{\partial^2 u_2}{\partial x^2}\right) = c_1\mathcal{L}(u_1)+c_2\mathcal{L}(u_2)$$

Definition: The PDE $\mathcal{L}(u)=f$ is linear if $\mathcal{L}$ is linear
- it is homogeneous if $f=0$
- it is non-homogeneous if $f\neq 0$.

From the definition of linearity, we can then deduce a fundamental property of homogeneous linear equations.

Principle of superposition: Given a homogeneous linear PDE $\mathcal{L}(u)=0$, if u_1 and u_2 are solutions of the PDE then $v=c_1u_1+c_2u_2$ is also a solution for any pair of constants c_1 and c_2

In other words, any linear combination of solutions is also a solution.

This principle is at the heart of the method of separation of variables.

Linearity & homogeneity of boundary conditions:

Boundary conditions can be recast as $\mathcal{B}_A(u)=g_A$ at $x=x_A$ and $\mathcal{B}_B(u)=g_B$ at $x=x_B$. If $\mathcal{B}_A$ & $\mathcal{B}_B$ are linear, then the boundary conditions are said to be linear. If $g_A=g_B=0$ then they are homogeneous.

II Separation of variables (for equations with 2 variables)

The following method can be applied to any linear, homogeneous equation with constant coefficients, with linear, homogeneous boundary conditions. Later on, we will see how to extend it to non-homogeneous problems with non-constant coeffs.

Here we apply it to the case of the heat equation in the following way.

(A) The mathematical & physical problem

Mathematical question

$$\begin{cases} \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} \\ \left.\begin{array}{l} u(0,t)=0 \\ u(L,t)=0 \end{array}\right] \rightarrow \text{BCs} \\ u(x,0) = f(x) \rightarrow \text{IC}. \end{cases}$$

This could correspond, for example, to the case of a metal (conducting) wire, insulated around the sides, with both ends held at temperature $T=0$ and with temperature profile $T(x) = f(x)$ at time $t=0$

(B) Separation of variables

The idea here is to find a general solution of the problem by constructing it as a linear superposition of basic functions which satisfy the PDE.
We look for basic functions of the kind

Step 1 Assume that

$$u(x,t) = A(x)B(t) \quad \leftarrow \text{ the variables are "separated".}$$

Step 2 Plugging back into the PDE we get

$$A(x)\frac{dB}{dt} = k\, B(t)\frac{d^2A}{dx^2}$$

Now divide by $A(x)B(t)$:

$$\rightarrow \quad \underbrace{\frac{1}{B}\frac{dB}{dt}}_{\text{A function of time only}} = k \underbrace{\frac{1}{A}\frac{d^2A}{dx^2}}_{\text{A function of space only.}}$$

Step 3 This equality has to hold for all time t and all position x, so the only way to satisfy it is to have both sides equal to a constant:

$$\frac{1}{B}\frac{dB}{dt} = \frac{k}{A}\frac{d^2A}{dx^2} = \lambda$$

λ is called the separation constant, so far is arbitrary.

Step 4 $\Rightarrow$ Now we have turned the original PDE problem into 2 ODEs, coupled by λ only.

$$\begin{cases} \dfrac{1}{B}\dfrac{dB}{dt} = \lambda \\ \dfrac{k}{A}\dfrac{d^2A}{dx^2} = \lambda \end{cases}$$

We can now analyse them one by one.

Note: Steps 1–4 are completely general & can be applied to a wide class of PDEs

(c) The solution of the spatial problem

The spatial problem is now an eigen-problem

$$\begin{cases} \dfrac{k}{A}\dfrac{d^2A}{dx^2} = \lambda \\ A(0)=0 \quad A(L)=0 \end{cases} \qquad \left(\text{since } \begin{cases} u(0,t)=0 \\ u(L,t)=0 \end{cases} \text{for all times}\right).$$

Note how λ has become an eigenvalue of the spatial problem, and A, the eigenfunction,

e.g.

$$\underbrace{k\frac{d^2A}{dx^2}}_{\text{operator on } A} = \lambda A$$

← eigenvalue times A.

Solutions of the problem depend on the sign of λ: there are 3 possibilities

1. $\lambda > 0$ then $A(x) = a_1 e^{\sqrt{\frac{\lambda}{k}}x} + a_2 e^{-\sqrt{\frac{\lambda}{k}}x}$

2. $\lambda = 0$ then $A(x) = a_1 + a_2 x$

3. $\lambda < 0$ then $A(x) = a_1 \cos\left(\sqrt{\frac{-\lambda}{k}}x\right) + a_2 \sin\left(\sqrt{\frac{-\lambda}{k}}x\right)$

Note how applying the boundary conditions implies that only the 3rd case will lead to a non-trivial (non-zero) solution. (check for yourself)

$$\Rightarrow \quad A(x) = a_1 \cos\left(\sqrt{\frac{-\lambda}{k}}x\right) + a_2 \sin\left(\sqrt{\frac{-\lambda}{k}}x\right)$$

is the only one that works.

$A(0)=0 \Rightarrow a_1 = 0$. $\quad A(L)=0 \Rightarrow a_2 \sin\left(\sqrt{\frac{-\lambda}{k}}L\right) = 0$.

Now we can't take $a_2 = 0$ (otherwise again we only get a trivial solution) $\rightarrow$ need $\sin\left(\sqrt{\frac{\lambda}{k}}L\right) = 0$

$\Rightarrow$ this determines λ, the eigenvalue:

$$\sqrt{\frac{-\lambda}{k}}\,L = n\pi \quad \text{where } n \text{ is an integer}$$

$$\Rightarrow \quad -\lambda_n = \frac{n^2\pi^2}{L^2}k.$$

$$\boxed{\lambda_n = -\frac{n^2\pi^2}{L^2}k}$$

Note that there is an ∞ # of possible eigenvalues

To each eigenvalue corresponds a particular eigenfunction

$$A_n(x) = \sin\left(\sqrt{\frac{-\lambda_n}{k}}x\right) = \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } n = 1 \ldots \infty.$$

$\Rightarrow$ We were looking for a solution to the spatial problem, and we found an infinite number of them, each a possible <u>eigenmode</u> of the spatial equation with its own eigenvalue. This is very typical of the method of separation of variables & we will see why later.

<u>Note</u>: As with all eigen-problems, the normalization of the eigenfunction becomes irrelevant (note how we dropped a_2). See more later.

D <u>The solution of the temporal problem(s)</u>

Given that we found an ∞ # of solutions for the spatial problem, each with its own λ_n, we will also have an ∞ of temporal solutions, each satisfying

$$\frac{1}{B_n}\frac{dB_n}{dt} = \lambda_n \quad \Rightarrow \quad \frac{dB_n}{dt} = -\frac{n^2\pi^2}{L^2}k\,B_n$$

Solutions are $B_n(t) = b_n e^{-\frac{n^2\pi^2}{L^2}kt}$

arbitrary constant

(E) The principle of superposition

We have now found an ∞ list of "basic solutions", each of the form

$$u_n(x,t) = A_n(x)B_n(t) = b_n e^{-\frac{n^2\pi^2}{L^2}kt} \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } n = 1 \ldots \infty$$

(see now why we can drop a_2? if we hadn't, it would just end up multiplying b_n in the $B_n(t)$ solution. Since b_n is arbitrary, it can be made to incorporate a_2)

$\Rightarrow$ Any linear combination of the $u_n(x,t)$ is also a solution of the PDE: so

$$\boxed{u(x,t) = \sum_n b_n e^{-\frac{n^2\pi^2}{L^2}kt} \sin\left(\frac{n\pi x}{L}\right)}$$

is also a solution (again, note how we can incorporate the arbitrary constant into b_n).

What we will show later is that this is in fact <u>the</u> most general form of the solution, i.e., that the actual solution of the PDE + associated conditions <u>has</u> to be of this form.

(F) Recap so far

So far we have obtained a general solution of the PDE which satisfies the equation and its (spatial) boundary conditions.

The only thing left to do is to make sure it also satisfies the initial condition (at $t=0$). From this, we will be able to deduce the value of the constants $\{b_n\}$.

(G) Some simple initial conditions

Example 1

Suppose that the initial condition to the problem is

$$u(x,0) = 10 \sin\left(\frac{4\pi x}{L}\right)$$

$\Rightarrow$ Then $10 \sin\left(\frac{4\pi x}{L}\right) = \sum_n b_n \sin\left(\frac{n\pi x}{L}\right)$ at $t=0$

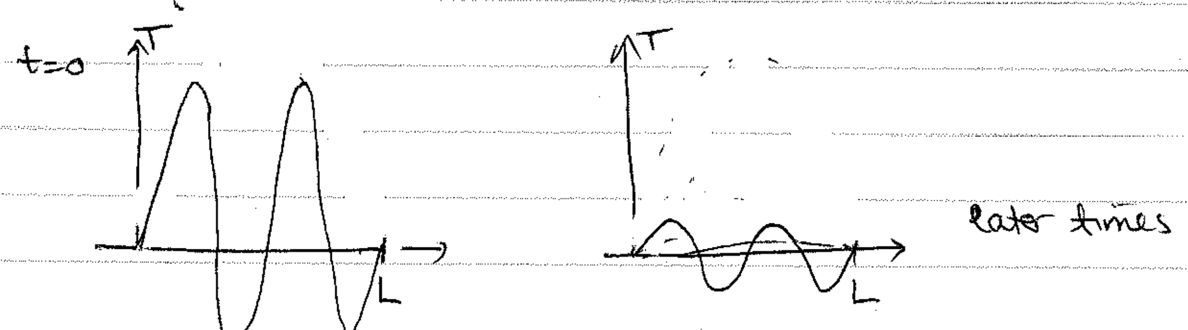
A simple way to get this is to choose $b_4 = 10$ and $b_n = 0 \quad \forall \quad n \neq 4$

$\rightarrow$ The solution to $\begin{cases} \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ with} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = 10 \sin\left(\frac{4\pi x}{L}\right) \end{cases}$

is

$$u(x,t) = 10 e^{-\frac{16\pi^2}{L^2}\cdot kt} \sin\left(\frac{4\pi x}{L}\right)$$

This is a sin function of exponentially decaying amplitude



Example 2 Suppose now that

$$u(x,0) = 2\sin\left(\frac{\pi x}{L}\right) + 100\sin\left(\frac{10\pi x}{L}\right)$$

Then, similarly, we find that the solution has to be such that $b_1 = 2$, $b_{10} = 100$ and $b_n = 0$ $\forall\, n \neq 1$ or 10. $\Rightarrow$

$$u(x,t) = 2e^{-\frac{\pi^2}{L^2}kt}\sin\left(\frac{\pi x}{L}\right) + 100\, e^{-\frac{100\pi^2}{L^2}kt}\sin\left(\frac{10\pi x}{L}\right).$$

<u>Important physical property of diffusion</u>

$\Rightarrow$ Small scales diffuse much faster than large scales.

Note how the rapidly oscillating $\sin\left(\frac{10\pi x}{L}\right)$ term has a decay rate of $\frac{100\pi^2 k}{L^2}$, while the slowly oscillating $\sin\left(\frac{\pi x}{L}\right)$ has a 100-times slower decay rate of $\frac{\pi^2 k}{L^2}$.

(see more)

(H) <u>Question</u>: What to do with more complicated initial conditions? ... (see next lecture)