CHAPTER 3 Second order linear PDEs : Canonical form

3.1 Definition

- A second order linear PDE has the general form

$$\mathcal{L}(u) = a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} \leftarrow \boxed{\text{principal part}}$$

$$+ \; d(x,y)u_x + e(x,y)u_y + f(x,y)u = g(x,y)$$

- The principal part is the part of the equation that only involves second-order derivatives.

- The discriminant of the linear operator $\mathcal{L}$ is defined as

$$\delta(\mathcal{L}) = b^2(x,y) - a(x,y)c(x,y)$$

and, in the general case, will be a function of x and y.

Examples

- the wave equation: $u_{tt} = c^2(x)u_{xx}$

write as $u_{tt} - c^2(x)u_{xx} = 0$

so $\delta(\mathcal{L}) = c^2(x)$

- the heat equation $u_t = ku_{xx}$

write as $ku_{xx} - u_t = 0$

so $\delta(\mathcal{L}) = -k \cdot 0 = 0$

- the Laplace equation: $u_{xx} + u_{yy} = 0$

$$\to \quad \delta(\mathcal{L}) = -1$$

<u>Definition</u>: An operator is hyperbolic / parabolic / elliptic at a point (x,y) if $\delta(\mathcal{L})$ is respectively >0, $=0$ or <0 at this point

The operator for the wave equation is hyperbolic at all points (assume $c^2(x) > 0$).

——— the heat equation is parabolic at all points

——— Laplace equation is elliptic at all points

$\Rightarrow$ An equation is hyperbolic / parabolic / elliptic in a domain D if its corresponding operator is hyperbolic / parabolic / elliptic at all points in D

3.2 Properties of the discriminant under a change of coordinate

The sign of the discriminant of an operator $\mathcal{L}$ is invariant under a <u>change of coordinates</u> from (x,y) to (ξ, η) (such that the Jacobian $J = \xi_x \eta_y - \xi_y \eta_x \neq 0$ for all (x,y)).

In other words, the type of an equation is an intrinsic property of the equation and is independent of the coordinate system in which the equation is written.

Proof Let $\xi = \xi(x,y)$
$\eta = \eta(x,y)$ $\Rightarrow$ now, $u(x,y) = w(\xi(x,y), \eta(x,y))$

then $$\frac{\partial u}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial w}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial w}{\partial \eta} \longrightarrow \frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}$$

$$\frac{\partial u}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial w}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial w}{\partial \eta} \longrightarrow \frac{\partial}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial \xi}{\partial x}\frac{\partial w}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial w}{\partial \eta}\right]$$

$$= \frac{\partial^2 \xi}{\partial x^2}\frac{\partial w}{\partial \xi} + \frac{\partial \xi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial \xi}\right) + \frac{\partial^2 \eta}{\partial x^2}\frac{\partial w}{\partial \eta} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial \eta}\right)$$

$$= \frac{\partial^2 \xi}{\partial x^2}\frac{\partial w}{\partial \xi} + \frac{\partial \xi}{\partial x}\left[\frac{\partial \xi}{\partial x}\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial \eta}{\partial x}\frac{\partial^2 w}{\partial \eta \partial \xi}\right]$$
$$+ \frac{\partial^2 \eta}{\partial x^2}\frac{\partial w}{\partial \eta} + \frac{\partial \eta}{\partial x}\left[\frac{\partial \xi}{\partial x}\frac{\partial^2 w}{\partial \xi \partial \eta} + \frac{\partial \eta}{\partial x}\frac{\partial^2 w}{\partial \eta^2}\right]$$

$$= \frac{\partial^2 \xi}{\partial x^2}\frac{\partial w}{\partial \xi} + \left(\frac{\partial \xi}{\partial x}\right)^2\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 \eta}{\partial x^2}\frac{\partial w}{\partial \eta} + \left(\frac{\partial \eta}{\partial x}\right)^2\frac{\partial^2 w}{\partial \eta^2} + 2\frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x}\frac{\partial^2 w}{\partial \xi \partial \eta}$$

and similarly

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}\frac{\partial^2 w}{\partial \eta^2} + \left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right)\frac{\partial^2 w}{\partial \xi \partial \eta}$$
$$+ \frac{\partial w}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial w}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \xi}{\partial y^2}\frac{\partial w}{\partial \xi} + \left(\frac{\partial \xi}{\partial y}\right)^2\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 \eta}{\partial y^2}\frac{\partial w}{\partial \eta} + \left(\frac{\partial \eta}{\partial y}\right)^2\frac{\partial^2 w}{\partial \eta^2}$$
$$+ 2\frac{\partial \eta}{\partial y}\frac{\partial \xi}{\partial y}\frac{\partial^2 w}{\partial \xi \partial \eta}$$

So the new PDE is

$$\tilde{L}(w) = A w_{\xi\xi} + 2B w_{\xi\eta} + C w_{\eta\eta} + D w_{\xi} + E w_{\eta} + F w = G$$

with
$$A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2$$
$$B = a\eta_x\xi_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\eta_y\xi_y$$
$$C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2$$

Another way of writing this is

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{pmatrix}$$

$$= J \begin{pmatrix} a & b \\ b & c \end{pmatrix} J^T$$

Now since $\delta(\mathcal{L}) = -\begin{vmatrix} a & b \\ b & c \end{vmatrix}$ then

$$\delta(\tilde{\mathcal{L}}) = -\begin{vmatrix} A & B \\ B & C \end{vmatrix} = +|J|\, \delta(\mathcal{L})\, |J^T| = |J||J^T|\, \delta(\mathcal{L}) = |J|^2\, \delta(\mathcal{L})$$

$\Rightarrow$ so provided $|J| \neq 0$ $\delta(\tilde{\mathcal{L}})$ has the same sign as $\delta(\mathcal{L})$, as required.

3.3 Canonical forms

We now consider three types of equations:
hyperbolic equations ($\delta(\mathcal{L}) > 0$ everywhere)
parabolic equations ($\delta(\mathcal{L}) = 0$ ————)
and elliptic equations ($\delta(\mathcal{L}) < 0$ ————)

It is possible to find a coordinate transform $(x, y) \to (\xi, \eta)$ reducing these equations to their <u>canonical forms</u> such that

($\delta(\tilde{\mathcal{L}}) = 1/4$) • hyperbolic equations become $u_{\xi\eta} + l_1(u) = g(\xi, \eta)$

($\delta(\tilde{\mathcal{L}}) = 0$) • parabolic equations ———— $u_{\xi\xi} + l_1(u) = g(\xi, \eta)$

($\delta(\tilde{\mathcal{L}}) = -1$) • elliptic equations ———— $u_{\xi\xi} + u_{\eta\eta} + l_1(u) = g(\xi, \eta)$

where $l_1(u)$ is a linear operator of first order.

3.3.1 Canonical form of Hyperbolic equations

Consider a hyperbolic eq. $\mathcal{L}(u) = a u_{xx} + 2b u_{xy} + c u_{yy} + d u_x + e u_y + f = g$
To change it into its canonical form we require a coordinate transform $(x, y) \to (\xi, \eta)$ such that

$$A = C = 0 \qquad \text{(in the notation of 3.2)}$$

$$\Leftrightarrow \begin{cases} a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0 \\ a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0 \end{cases}$$

$\to$ two equations are equivalent

Now rewrite $a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = a\left[\xi_x^2 + \frac{2b}{a}\xi_x\xi_y + \frac{c}{a}\xi_y^2\right]$

provided $a \neq 0$ $= a\left[\left(\xi_x + \frac{b}{a}\xi_y\right)^2 + \frac{c}{a}\xi_y^2 - \frac{b^2}{a^2}\xi_y^2\right]$

$$= a\left[\left(\xi_x + \frac{b}{a}\xi_y\right)^2 - \frac{b^2}{a^2}\xi_y^2\left(1 - \frac{c}{a}\frac{a^2}{b^2}\right)\right]$$

$$= a\left[\left(\xi_x + \frac{b}{a}\xi_y\left(1 + \sqrt{1 - \frac{ca}{b^2}}\right)\right) \cdot \left(\xi_x + \frac{b}{a}\xi_y\left(1 - \sqrt{1 - \frac{ca}{b^2}}\right)\right)\right]$$

So $a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0$ if and only if

$$\xi_x + \frac{b}{a}\left(1 + \sqrt{1 - \frac{ca}{b^2}}\right)\xi_y = 0$$

OR

$$\xi_x + \frac{b}{a}\left(1 - \sqrt{1 - \frac{ca}{b^2}}\right)\xi_y = 0.$$

$\to$ Let's choose ξ a solution of $\xi_x + \frac{b}{a}\left(1 + \sqrt{1 - ca/b^2}\right)\xi_y = 0$

η ——— $\eta_x + \frac{b}{a}\left(1 - \sqrt{1 - ca/b^2}\right)\eta_y = 0$

- ξ is constant on the characteristics defined from

$$\frac{dx}{d\tau} = 1 \qquad \frac{dy}{d\tau} = \frac{b}{a}\left(1 + \sqrt{1 - ca/b^2}\right)$$

or $$dy/dx = \frac{b}{a}\left(1 + \sqrt{1 - ca/b^2}\right) = \frac{b + \sqrt{b^2 - ac}}{a}$$

and η is constant on characteristics satisfying

$$\frac{dy}{dx} = \frac{b}{a}\left(1 - \sqrt{1 - ac/b^2}\right) = \frac{b - \sqrt{b^2 - ac}}{a}$$

3.3.2 Examples

① The wave equation

$$u_{tt} - c^2 u_{xx} = 0 \qquad \delta(\mathcal{L}) = c^2 > 0$$

$\to$ a hyperbolic equation

To find a coordinate system (ξ, η) in which the wave equation is reduced to its canonical form, we must solve

$$\xi_t^2 - c^2 \xi_x^2 = 0$$

$$\Leftrightarrow (\xi_t - c\xi_x)(\xi_t + c\xi_x) = 0$$

let ξ be the solution of $\xi_t - c\xi_x = 0$

η ——————— $\eta_t + c\eta_x = 0$

ξ is constant on characteristics satisfying $\frac{dx}{dt} = -c$

$\Leftrightarrow \quad x = -ct + \text{constant}$

Let ξ be that constant, so that $\xi(x,t) = x + ct$

η is constant on characteristics satisfying $\frac{dx}{dt} = c$

$\Leftrightarrow \quad x = ct + \text{constant}$

Let η be that constant, so that $\eta(x,t) = x - ct$.

In the new coordinate system, we verify that

$$u_{tt} - c^2 u_{xx} = -u_{\xi\eta} \cdot 4c^2 = 0$$

Indeed $\quad u_t = cu_\xi - cu_\eta \qquad u_{tt} = c^2 u_{\xi\xi} + c^2 u_{\eta\eta} - 2c^2 u_{\eta\xi}$

$\quad u_x = u_\xi + u_\eta \qquad u_{xx} = u_{\xi\xi} + u_{\eta\eta} + 2u_{\eta\xi}$

$\Rightarrow$ So $\quad -2c^2 u_{\eta\xi} - 2c^2 u_{\eta\xi} = 0 \Rightarrow \boxed{u_{\eta\xi} = 0.}$

We can now find the solutions straightforwardly:

$$u_{\xi\eta} = 0 \iff u = F(\xi) + G(\eta)$$
$$= F(x+ct) + G(x-ct)$$

where F and G are chosen to satisfy the required boundary conditions.

SKIP ② The Tricomi equation

$$u_{xx} + x u_{yy} = 0 \qquad \delta(\mathcal{L}) = -x$$

so the equation is hyperbolic for $x<0$.

→ We restrict the following work to the $x<0$ domain.

We seek the change of variable $(x,y) \to (\xi,\eta)$ which will simplify it into a canonical form

⇒ we require

$$\xi_x^2 + x\,\xi_y^2 = 0 = \xi_x^2 - |x|\,\xi_y^2$$

$$\iff \left(\xi_x + \sqrt{|x|}\,\xi_y\right)\left(\xi_x - \sqrt{|x|}\,\xi_y\right) = 0$$

let ξ be the solution of $\xi_x + \sqrt{|x|}\,\xi_y = 0$

→ ξ is constant on characteristics determined from

$$\frac{dy}{dx} = \sqrt{|x|}$$

$$\iff y = \frac{2}{3}|x|^{\frac{3}{2}} + \text{constant}$$

so $\xi = y - \frac{2}{3}|x|^{\frac{3}{2}}$

Similarly for η, $\frac{dy}{dx} = -\sqrt{|x|}$ so

$$y = -\frac{2}{3}|x|^{\frac{3}{2}} + \text{constant}$$

$$\Rightarrow \eta = y + \frac{2}{3}|x|^{\frac{3}{2}}$$

then $\quad \xi_y = 1, \quad \xi_x = -|x|^{\frac{1}{2}} \qquad \xi_{yy} = 0 \qquad \xi_{xx} = -\dfrac{1}{2|x|^{1/2}}$

$\quad \eta_y = 1 \qquad \eta_x = |x|^{1/2} \qquad \eta_{yy} = 0 \qquad \eta_{xx} = \dfrac{1}{2|x|^{1/2}}.$

so $\quad u_x = \xi_x u_\xi + \eta_x u_\eta \qquad\qquad u_y = u_\xi + u_\eta$

$$= -|x|^{1/2} u_\xi + |x|^{1/2} u_\eta$$

$$u_{xx} = -\tfrac{1}{2}|x|^{-\frac{1}{2}} u_\xi + \tfrac{1}{2}|x|^{-\frac{1}{2}} u_\eta - |x|^{\frac{1}{2}}\left[\xi_x u_{\xi\xi} + \eta_x u_{\xi\eta}\right]$$
$$+ |x|^{1/2}\left[\xi_x u_{\xi\eta} + \eta_x u_{\eta\eta}\right]$$
$$= |x|(u_{\xi\xi} + u_{\eta\eta}) - |x| u_{\xi\eta} + \tfrac{1}{2}|x|^{-\frac{1}{2}}(u_\eta - u_\xi)$$

$$u_{yy} = \left[\xi_y u_{\xi\xi} + \eta_y u_{\xi\eta} + \xi_y u_{\eta\xi} + \eta_y u_{\eta\eta}\right]$$
$$= u_{\xi\xi} + u_{\eta\eta} + 2u_{\eta\xi}$$

so

$$|x|(u_{\xi\xi} + u_{\eta\eta}) - |x| u_{\xi\eta} + \tfrac{1}{2}|x|^{-\frac{1}{2}}(u_\eta - u_\xi) - |x|(u_{\xi\xi} + u_{\eta\eta} + 2u_{\eta\xi})$$
$$= -3|x| u_{\xi\eta} + \tfrac{1}{2}|x|^{-\frac{1}{2}}(u_\eta - u_\xi) = 0$$
$$\iff -3|x|^{\frac{3}{2}} u_{\xi\eta} + \tfrac{1}{2}(u_\eta - u_\xi) = 0$$

but $\quad |x|^{3/2} = \frac{3}{4}(\eta - \xi)$

so finally $\quad u_{\xi\eta} + \dfrac{4}{9}\,\dfrac{1}{\xi - \eta}\cdot\dfrac{1}{2}(u_\eta - u_\xi) = 0$

is the canonical form of the Tricomi equation