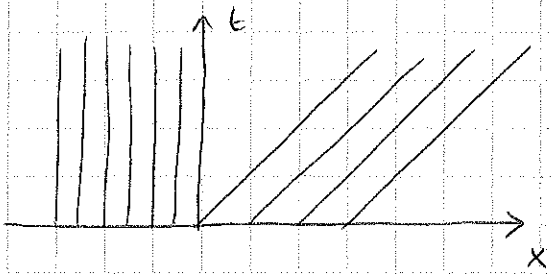
2.6.6 Expansion shocks and entropy condition

Consider the example

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = \Theta(x) \end{cases} \qquad \left(F(u) = \tfrac{1}{2}u^2\right) \quad \leftarrow \text{Heaviside function: } \begin{cases} \Theta(x) = 0 & x<0 \\ \Theta(x) = 1 & x \geq 0 \end{cases}$$

Characteristics



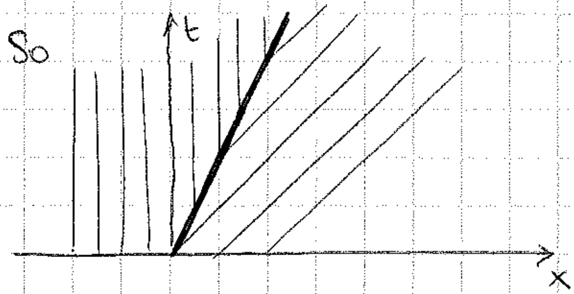
$$\begin{cases} t = \tau \\ x = u_0(s)\tau + s \\ u = u_0(s) = \Theta(s) \end{cases}$$

We could consider constructing a weak solution $u = u_- = 0$ on the left of $\gamma(t)$ and $u = u_+ = 1$ on the right of $\gamma(t)$.

The R.H. condition implies $\dfrac{d\gamma}{dt} = \dfrac{F(u_+) - F(u_-)}{u_+ - u_-} = \dfrac{1}{2}$

$\Rightarrow$ here $\gamma(t) = \frac{1}{2}t$

So



$$\begin{cases} u = 0 & x \leq t/2 \\ u = 1 & x > t/2 \end{cases}$$

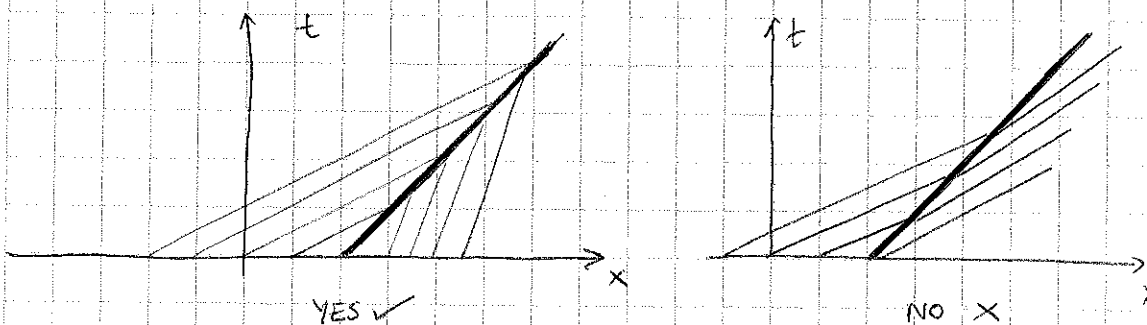
Problem: this solution is not physically acceptable because it is not causal: information appears to be "created" on the discontinuity and is then carried by the characteristics.

In other words: we would like the system to be entirely determined by its initial conditions, not by arbitrary extensions of the solution

Definition The entropy condition

Characteristics must enter the discontinuity (the shock front) but are not allowed to emanate from it

To guarantee this, the slope of the characteristics on the left must be shallower than $\delta(t)$, and those on the right steeper:



$$\Rightarrow \quad \frac{1}{F'(u_-)} \le \frac{1}{\frac{d\delta}{dt}} \le \frac{1}{F'(u_+)}$$

$$\Leftrightarrow \quad \boxed{F'(u_-) \ge \frac{d\delta}{dt} \ge F'(u_+)}$$

Problem: How do we construct solutions if $F'(u)$ is an increasing function of u? (see example above).

Going back to the characteristic solutions:

$$\begin{cases} t = \tau \\ x = F'(\phi(s))t + s \\ u = \phi(s) \end{cases} \quad \text{to} \quad \begin{cases} u_t + [F(u)]_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

Assume characteristics diverge from $s = s_0$. At this point,

$$x = F'(\phi(s_0))t = F'(u)t + s_0$$

so let's construct

$$u = G\left(\frac{x - s_0}{t}\right)$$

where G is the inverse function of F'.

and use this as a solution in the "fan" region

Example 1 $\quad u_t + uu_x = 0 \qquad F(u) = \frac{1}{2}u^2 \quad F'(u) = u$

with $\quad u(x,0) = \Theta(x)$

Characteristics diverge from $s=0$, so that

$$x = ut \quad \text{or} \quad u = x/t \quad \text{in the "fan"}$$

$\Rightarrow$ we construct the weak solution with

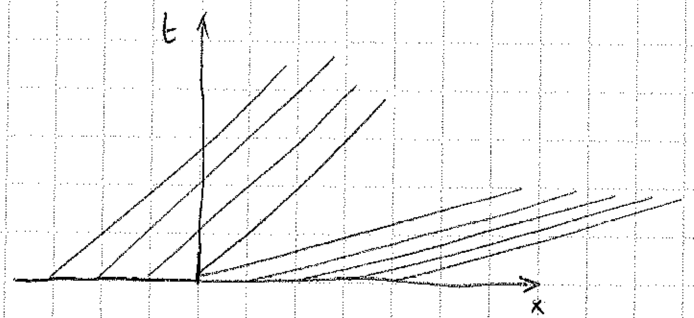
$$\begin{cases} u = 0 & \text{if } x \leq 0 \\ u = x/t & \text{if } 0 \leq x \leq t \\ u = 1 & \text{if } x \geq t \end{cases}$$

Example 2: $\quad u_t + (e^u)_x = 0 \quad$ with $\quad u(x,0) = \Theta(x)$

then $\qquad u_t + e^u u_x = 0$

$\rightarrow$ Characteristics are

$$t = \frac{x-s}{e^{\Theta(s)}} = \begin{cases} \frac{x-s}{1} & \text{if } s \leq 0 \\ \frac{x-s}{e} & \text{if } s \geq 0 \end{cases}$$



So let's construct from $x = e^u t + s$ the solution

$u = \ln(x/t)$ emanating from $s=0$

$$\Rightarrow \begin{cases} u = 0 & \text{if } x \leq t \\ u = \ln(x/t) & \text{if } t \leq x \leq et \\ u = 1 & \text{if } x \geq et \end{cases}$$

Check: in region $\quad t \leq x \leq et$

$$\frac{\partial u}{\partial t} = -\frac{x}{t^2}\cdot\frac{t}{x} = -\frac{1}{t} \qquad e^u \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(e^u) = \frac{\partial}{\partial x}\left(\frac{x}{t}\right) = \frac{1}{t}$$

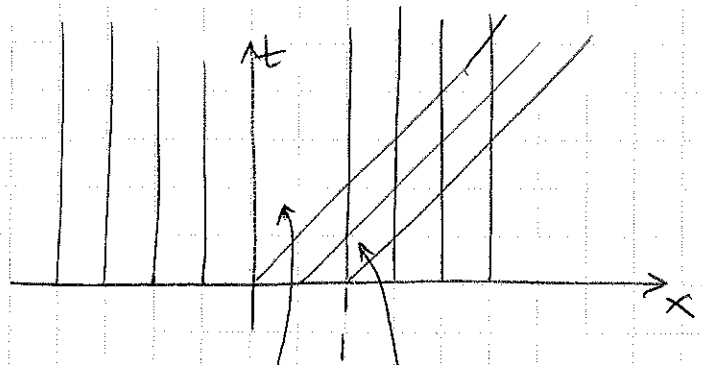
$$\Rightarrow \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(e^u) = 0 \text{ as required}$$

Problem with a shock & an expansion shock (midterm 2006)

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \begin{cases} 0 & \text{for } x \le 0 \\ 1 & \text{for } 0 < x < 1 \\ 0 & \text{for } x \ge 1 \end{cases} \end{cases} \qquad F(u) = \frac{u^2}{2}$$

- Characteristics: $\quad x = u_0(s)\,t + s$

  or $\quad t = \dfrac{x-s}{u_0(s)}$



- Solution in the fan region: $(0 \le x \le t)$

  In this region, all characteristics emerge from $s=0$ and $\quad x = ut \quad \Rightarrow u = \dfrac{x}{t}$.

- The initial shock position is at $\quad x_c = 1,\ t_c = 0$

- For $t \le T$, the shock occurs between characteristics emerging from the interval $[0,1]$ "carrying" the solution $u=1$, and from characteristics emerging from the interval $[1,+\infty]$ "carrying" $u=0$

T to be determined

so

$$u_+ = 0 \qquad \frac{ds}{dt} = \frac{\frac{1}{2}}{1} = \frac{1}{2}.$$
$$u_- = 1$$

so $\quad \gamma(t) = \frac{1}{2}t + \text{constant}$

Applying the boundary conditions we get

$$\boxed{\gamma(t) = \frac{1}{2}t + 1} \quad \text{for } t \leqslant 1.$$

At $t = T$, therefore the shock is at position $\gamma = x = T$

$$x = \frac{1}{2}T + 1 \quad \Rightarrow \quad T = \frac{1}{2}T + 1 \quad \Rightarrow \quad T = 2$$
$$\gamma(T) = 2$$

- for $t \geq T$ the shock is between characteristics emerging from the fan region and from the interval $[1, +\infty)$.

so $\quad u_- = \frac{x}{t} \qquad u_+ = 0$

On the shock front, $x = \gamma$ so $u_- = \gamma/t$.

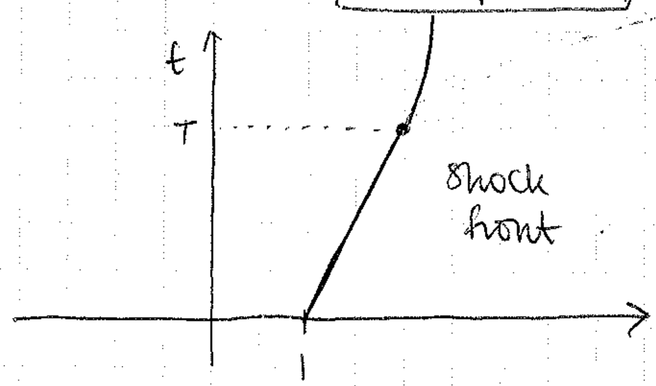
$$\Rightarrow \quad \frac{d\gamma}{dt} = \frac{-\frac{1}{2}\left(\frac{\gamma}{t}\right)^2}{0 - \frac{\gamma}{t}} = \frac{1}{2}\frac{\gamma}{t}$$

therefore $\quad \ln \gamma = \frac{1}{2}\ln t + \text{constant}$

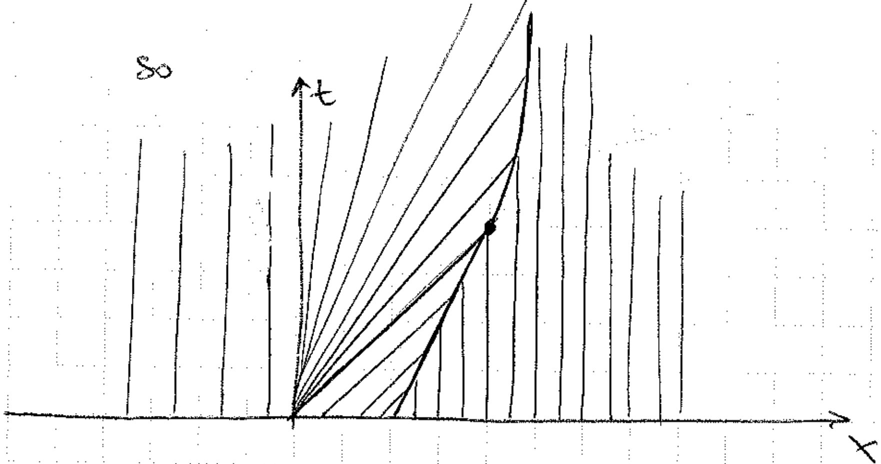
The "initial condition" to apply here is $\gamma =$ when $t = 2$ so

$$\ln(\gamma) = \frac{1}{2}\ln(t) + \frac{1}{2}\ln(2)$$

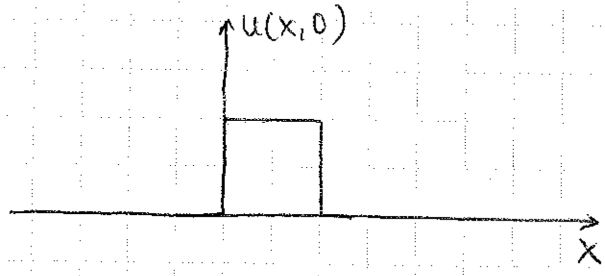
so $\quad \boxed{\gamma = \sqrt{2t}}$



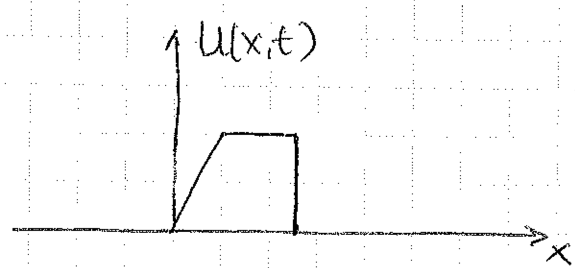
So



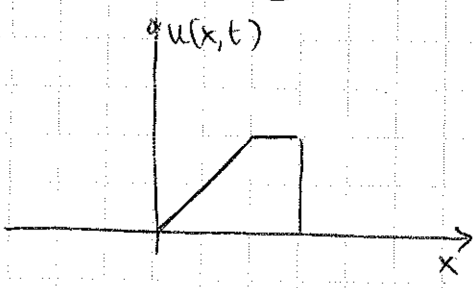
so at $t=0$



$0 < t < T$



t slightly bigger but $< T$



$t > T$