2.4 Examples of use of quasilinear 1st order PDES

2.4.1 Conservation laws (general)

- Conservation laws are usually written in the form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(F(u,x,t)\right) = 0$$

(or, in more than one dimension

$$\frac{\partial u}{\partial t} + \nabla\cdot\left(F(u,\underline{r},t)\right) = 0 \quad).$$

- Why is this called a conservation law?

⇒ Expressed in their integral form, the conservation laws describes the conservation of the quantity u:

$$\int_D \frac{\partial u}{\partial t}\, d^3\underline{r} + \int_D \nabla\cdot\left(F(u,\underline{r},t)\right) d^3\underline{r} = 0$$

$$\Leftrightarrow \quad \frac{\partial}{\partial t}\int_D u\, d^3\underline{r} + \int_{S = \text{contour of } D} F(u,\underline{r},t)\, d^2\underline{r} = 0$$

total change of the quantity u in the domain D

Flux through the surface of the domain

Example: The standard equation for the conservation of mass in a flow stirred by a velocity field $v(\underline{r})$ is

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\, v(\underline{r})\right) = 0$$

It can be rewritten as

$$\int_D \frac{\partial \rho}{\partial t} d\underline{r}^3 + \int_D \nabla \cdot (\rho v(\underline{r}))\, d^3\underline{r}$$

$$\Leftrightarrow \quad \frac{\partial m}{\partial t} + \int_{\substack{\text{surface} \\ \text{of } D}} \rho v(\underline{r})\, d^2\underline{r} = 0$$

change of mass within volume D

mass carried by velocity field across surface of D

- Conservation laws occur in most domains in science.

2.4.2 Conservation law (specific)

Here for simplicity we consider conservation laws which can be written as

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(F(u)\right) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

(the flux is a function of u only)

Then we see that

$$\frac{\partial u}{\partial t} + F'(u)\frac{\partial u}{\partial x} = 0$$

$\rightarrow$ a quasilinear homogeneous PDE that can be integrated with

$$\begin{cases} \frac{\partial t}{\partial \tau} = 1 & \longrightarrow \quad t = \tau \\ \frac{\partial x}{\partial \tau} = F'(u) & \qquad (*) \\ \frac{\partial u}{\partial \tau} = 0 & \rightarrow \quad u \text{ is constant along characteristics:} \end{cases}$$

$$u = v_0(s) = \phi(s)$$

So from (*) we see that

$$\frac{\partial x}{\partial \tau} = F'(\phi(s)) \longrightarrow x = F'(\phi(s))\tau + x_0(s)$$

in other words

$$\left\{\begin{array}{l} \tau = t \\ x = F'(\phi(s))t + s \\ u = \phi(s) \end{array}\right\} \text{ so } \quad u = \phi(x - F'(u)t)$$

<u>Interpretation</u>

① u is constant on characteristics

② The characteristics are <u>straight lines</u> in the $(x\text{-}t)$ plane with slope $\dfrac{1}{F'(\phi(s))}$, which depends only on the initial condition (and on F!)

③ The problem of finding $u(x,t)$ becomes equivalent to solving the algebraic equation

$$u = \phi\left(x - F'(u)t\right)$$

It is sometimes possible to invert this analytically.

2.4.3 Burger's inviscid equation (Euler's equation)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

$$\Leftrightarrow \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \quad \rightarrow \text{ a conservation law with } F(u) = \frac{u^2}{2}$$

$$F'(u) = u$$

so the solution for any given initial condition $u(x,0) = \phi(x)$ can formally be written as

$$u(x,t) = \phi\left(x - u(x,t)\,t\right)$$

which may or may not be solvable analytically

① Suppose

$$\phi(x) = 3x \quad \text{then}$$

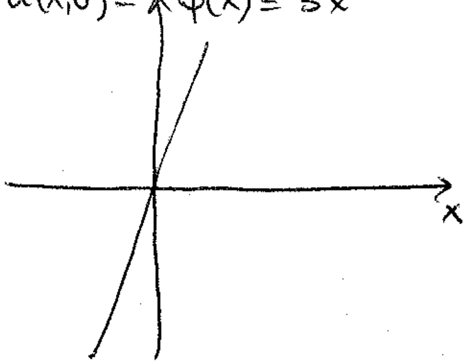
$$u(x,t) = 3\left(x - u(x,t)t\right)$$

$$\Leftrightarrow \quad u(x,t)\left[1 + 3t\right] = 3x$$

$$\Rightarrow \quad u(x,t) = \frac{3x}{1+3t}$$

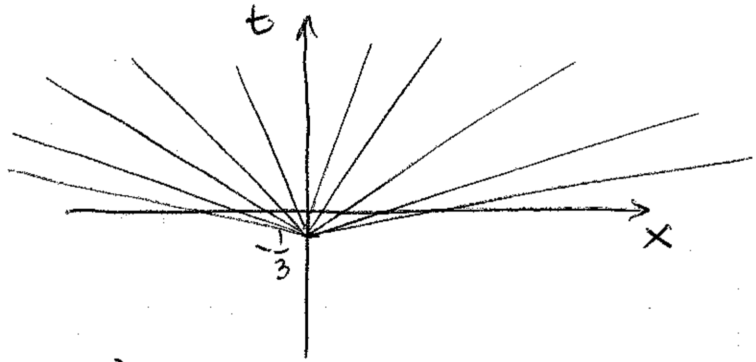
<u>Interpretation</u>

$u(x,0) = \phi(x) = 3x$



Characteristics are lines in the $(x\text{-}t)$ plane with slope $\frac{1}{F'(\phi(s))}$

Here the slope is $\frac{1}{3s}$



$$C^{(s)} : x = 3s \cdot t + s$$

$$\Leftrightarrow \quad t = \frac{x-s}{3s}$$

(they all pass through the point $\left(0, -\frac{1}{3}\right)$)

$u(x,t)$ is constant along a characteristic

② Suppose $\phi(x) = e^{-x^2/2}$

then we have to invert

$$u(x,t) = e^{-\left(x - u(x,t)t\right)^2/2}$$

to find $u(x,t) \rightarrow$ difficult.

2.5 Weak solutions, shocks and entropy condition

2.5.1 Example of Burger's equation

$u_t + u u_x = 0$
$u(x,0) = f(x)$

Characteristic equations:

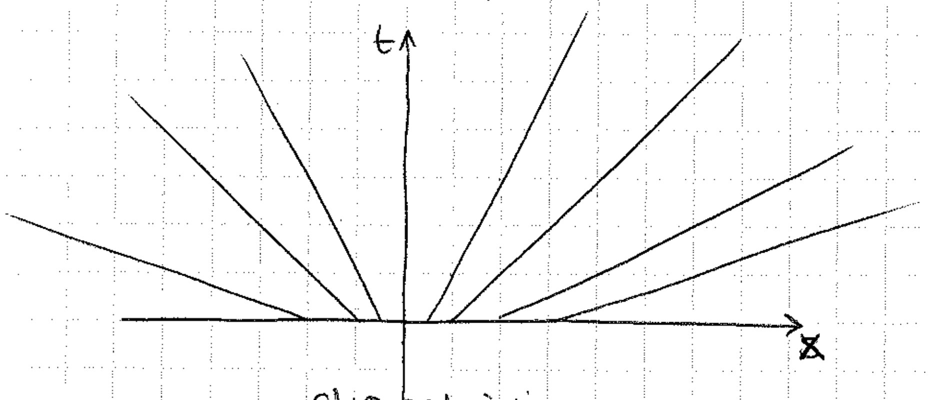
$\frac{dt}{d\tau} = 1 \rightarrow t = \tau$

$\frac{dx}{d\tau} = u \rightarrow x = u\tau + s$

$\frac{du}{d\tau} = 0 \rightarrow u = u_0(s) = f(s)$

Characteristics: Straight lines $t = \frac{x-s}{f(s)}$

Example 1: $f(s) = s$



Characteristics

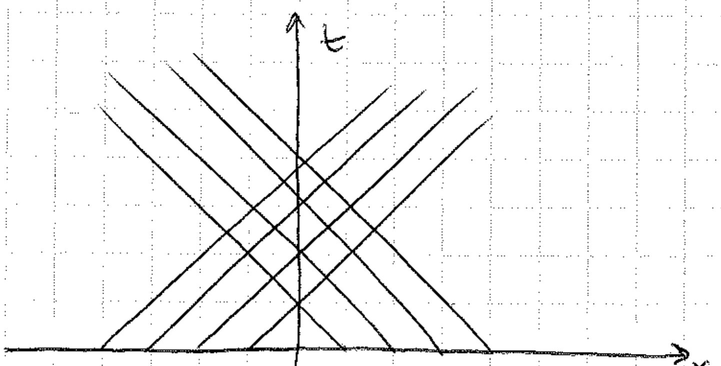
→ the solution exists at all times, no problem

$u = f(s) = s$
$\quad = x - ut$

so $u = \frac{x}{1+t}$

Example 2: First type of problem crossing characteristics

$$f(s) = -\frac{|s|}{s} = \begin{cases} 1 & \text{if } x < 0 \\ -1 & \text{if } x \geq 0 \end{cases}$$

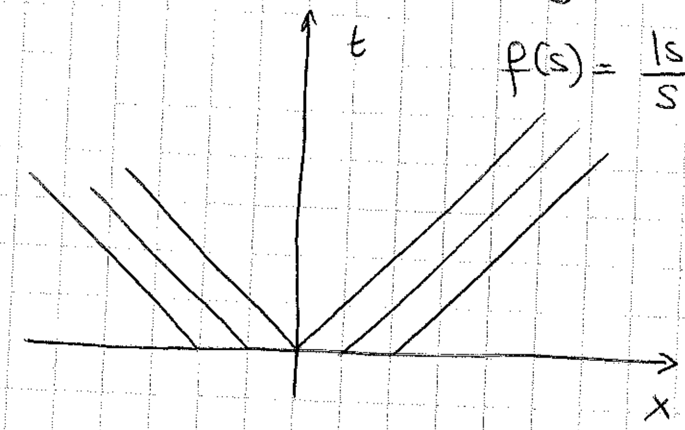


Characteristics intercect!

Since $u = u_0(s)$ is constant on characteristics, which value should we choose?!

Example 3 : Second type of problem: some region of space/time is not represented by any characteristic.

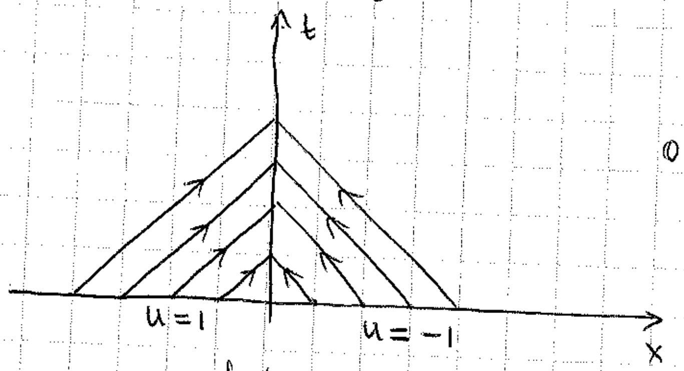
$$f(s) = \frac{|s|}{s} = \begin{cases} -1 & \text{if } x<0 \\ 1 & \text{if } x \geq 0 \end{cases}$$



What values should $u(x,t)$ take in the region which is not spanned by any characteristics?

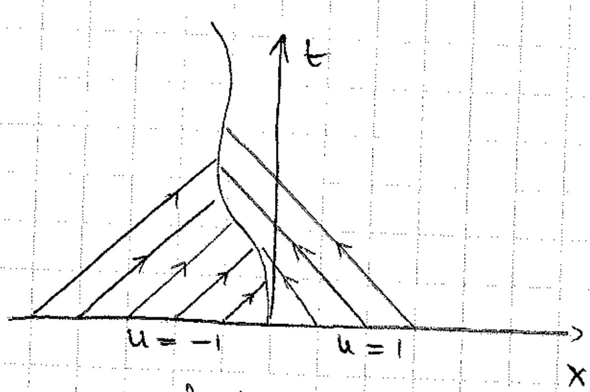
2.5.2 Weak problems and weak solutions

One way to resolve the first type of problem is to select a particular line separating characteristics emanating from the left & from the right and selecting the corresponding solution on each side



selecting x = 0 line

or



selecting a random line ...

Problems!
- the solution then appears to be discontinuous along the line ⇒ SHOCK
- there is more than one solution

Non-smooth solutions are called weak solutions. Weak solutions are not solutions of the PDE since $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ are not defined at discontinuities.
Weak solutions are solutions of the associated weak problem

<u>Definition</u>: A weak problem is an <u>integral</u> reformulation of the PDE for which solutions can be discontinuous!

<u>Note</u>: there are many possible weak problems associated to a given PDE.

2.5.3 Weak problems and conservation laws

Conservation laws of the kind

$$\frac{\partial u}{\partial t} + \nabla \cdot F = 0$$

are usually derived in physical systems from integral relationships anyway →

$$\frac{\partial}{\partial t} \int_{\text{volume}} u \, dV + \int_{\text{surface}} F \cdot dS = 0 = \frac{\partial}{\partial t} \int_{\text{volume}} u \, dV + \int_{\text{volume}} \nabla \cdot F \, dV$$

so we may as well use these integral formulations as our weak problem.

Take $\quad u_t + \frac{\partial}{\partial x} \left[F(u) \right] = 0$

and integrate over an interval $[a, b]$ at a given time t:

$$\frac{\partial}{\partial t} \int_a^b u \, dx + \int_a^b \frac{\partial}{\partial x} \left[F(u) \right] dx = 0$$

$$\Leftrightarrow \quad \frac{\partial}{\partial t} \int_a^b u \, dx + F(u(b,t)) - F(u(a,t)) = 0 \qquad (*)$$

→ this is the weak formulation of a conservation law

- Any smooth solution of $(*)$ is also a solution of the associated PDE.
- However, we can now construct non-smooth solutions.

Assume the solution has one discontinuity in the solution $u(x,t)$ located on the line $x = \gamma(t)$ such that

$$\begin{cases} u(x,t) = u_-(x,t) & \text{if} \quad x < \gamma(t) \\ u(x,t) = u_+(x,t) & \text{if} \quad x > \gamma(t) \end{cases}$$

Then, plugging this into (*) we get

$$\frac{\partial}{\partial t}\left[\int_a^{\gamma(t)} u_-(x,t)\,dx + \int_{\gamma(t)}^b u_+(x,t)\,dx\right] + F(u(b,t)) - F(u(a,t)) = 0$$

Recall: $$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = \frac{db}{dt} f(b(t),t) - \frac{da}{dt} f(a(t),t) + \int_{a(t)} \frac{\partial f}{\partial t}\,dx$$

So

$$\frac{\partial}{\partial t}\int_a^{\gamma(t)} u_-(x,t)\,dx = \frac{d\gamma}{dt} \underline{u}(\gamma(t),t) + \int_a^{\gamma} \frac{\partial u_-}{\partial t}\,dx$$

$$\frac{\partial}{\partial t}\int_{\gamma(t)}^b u_+(x,t)\,dx = -\frac{d\gamma}{dt} u_+(\gamma(t),t) + \int_{\gamma(t)}^b \frac{\partial u_+}{\partial t}\,dx$$

So $$\frac{d\gamma}{dt}\left[u_-(\gamma(t),t) - u_+(\gamma(t),t)\right] + \int_a^{\gamma(t)} \frac{\partial u_-}{\partial t}\,dx + \int_{\gamma(t)}^b \frac{\partial u_+}{\partial t}\,dx$$

$$+ F(u(b,t)) - F(u(a,t)) = 0.$$

Now write

$$F(u(b,t)) - F(u(a,t)) = F(u(b,t)) - F(u_+(\gamma(t),t)) + F(u_+(\gamma(t),t))$$
$$+ F(u_-(\gamma(t),t)) - F(u(a,t)) - F(u_-(\gamma(t),t))$$
$$= \int_a^{\gamma(t)} \frac{\partial}{\partial x}(F(\underline{u}))\,dx + \int_{\gamma(t)}^b \frac{\partial}{\partial x}(F(\underline{u}))\,dx + F(u_+(\gamma(t),t))$$
$$- F(u_-(\gamma(t),t))$$

So finally we get

$$\int_a^{\gamma(t)} \cancel{\frac{\partial u_-}{\partial t} + \frac{\partial}{\partial x}\left(F(u_-)\right)}\, dx + \int_{\gamma(t)}^{b} \cancel{\frac{\partial u_+}{\partial t} + \frac{\partial}{\partial x}\left(F(u_+)\right)}\, dx$$

$$+ \frac{d\gamma}{dt}\left(u_-(\gamma(t),t) - u_+(\gamma(t),t)\right) + F(u_+(\gamma(t),t)) - F(u_-(\gamma(t),t)) = 0$$

$$\Rightarrow \quad \frac{d\gamma}{dt} = \frac{F(u_+(\gamma(t),t)) - F(u_-(\gamma(t),t))}{u_+(\gamma(t),t) - u_-(\gamma(t),t)}$$

An equation for the discontinuity curve (shock curve) in terms of the jump in u and $F(u)$ across the shock. Sometimes written as

$$\frac{d\gamma}{dt} = \frac{[F]}{[u]}$$

Rankine – Hugoniot Jump condition.

To find $\gamma(t)$ we need an initial condition: take $\gamma(t_c) = x_c$ where t_c is the earliest time (with $t_c > 0$) for which characteristics cross and x_c is the position at which this happens.

<u>Example 1</u> Burgers equation $\left(F(u) = \frac{u^2}{2}\right)$ with $f(s) = -\frac{|s|}{s}$

We saw the earliest (positive) characteristics crossing occurs at $x_c = 0$, $t_c = 0$

On the left side of $\gamma(t)$, $u = u_- = 1$

— right —— $u = u_+ = -1$

$F(u_+) = \frac{1}{2}$ $\quad F(u_-) = \frac{1}{2}$ so

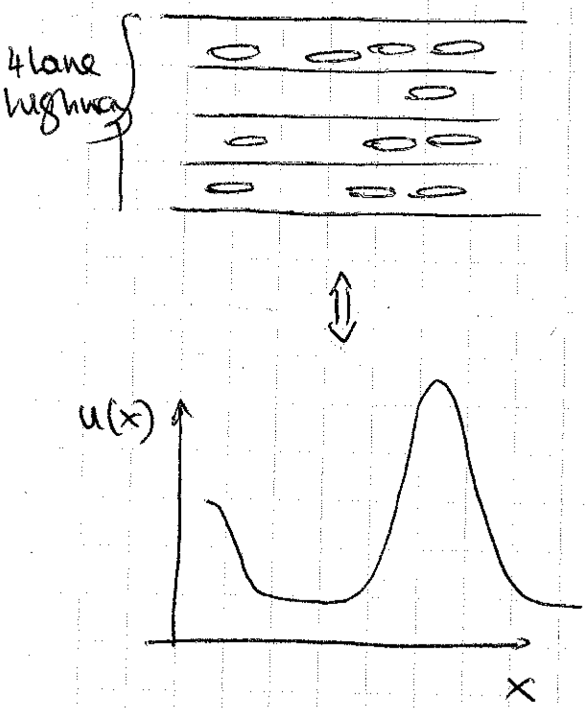
$$\frac{d\gamma}{dt} = \frac{0}{2} = 0 \Rightarrow \gamma = \text{constant} \Rightarrow \gamma = 0$$

So the correct discontinuity line is $\boxed{x = 0}$.

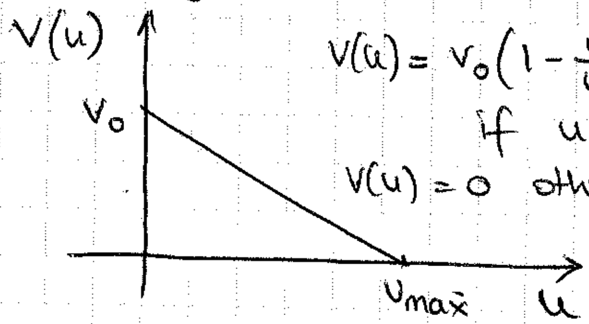
2.4.4 Traffic flow

The study of traffic flow is an attempt at modelling (for example) the flow of cars on a road / highway, but also for example of information on a network, etc ..

Idea: ① Model the road / network as a 1D line, with some density $u(x,t)$ of traffic (cars / information packets) at time t, position x.



② Model the velocity of the traffic flow as a function of the traffic density:



$$V(u) = V_0\left(1 - \frac{u}{U_{max}}\right) \quad \text{if } u \in [0, U_{max}]$$

$$V(u) = 0 \quad \text{otherwise}$$

→ Flowing traffic has optimal velocity V_0 when u is small, and stalls when $U > U_{max}$

The conservation law for the car density $u(x,t)$ is simply

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(u V(u)\right) = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[V_0 u\left(1 - \frac{U}{U_{max}}\right)\right] = 0$$

So here we have a conservation law with

$$F(u) = V_0 u\left(1 - \frac{U}{U_{max}}\right) \quad \text{if } 0 < u < U_{max}$$

$$= 0 \quad \text{otherwise}$$

$$\Rightarrow \quad F'(u) = V_0\left(1-\frac{u}{U_{max}}\right) - \frac{V_0 u}{U_{max}}$$

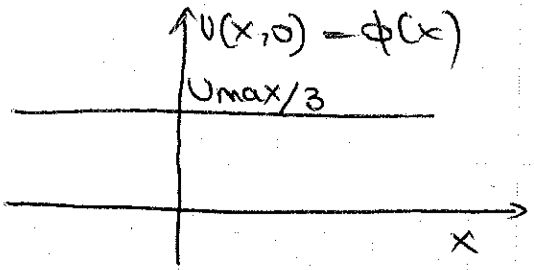
$$F'(u) \begin{cases} = V_0\left(1-\frac{2u}{U_{max}}\right) & \text{if} \quad u \in [0, U_{max}] \\ = 0 & \text{otherwise} \end{cases}$$

- The solution to any initial traffic condition $u(x,0) = \phi(x)$ is given by the algebraic equation

$$u(x,t) = \phi\left(x - F'(u)t\right)$$

- The solution $u(x,t)$ is constant along characteristics, which are straight lines with slope $\dfrac{1}{F'(\phi(s))}$

Example 1 Suppose we start with a uniform density of cars $u(x,0) = \frac{U_{max}}{3} \quad \forall x$.



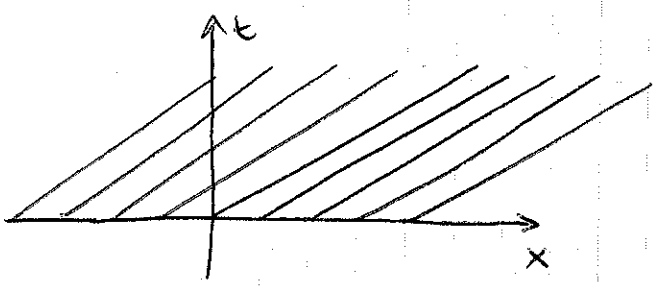
so $\phi(s) = \frac{U_{max}}{3} \quad \forall s$.

The characteristics are straight lines with equation

$$x = F'(\phi(s))t + s$$

$$\Leftrightarrow \quad x = F'\left(\frac{U_{max}}{3}\right)t + s$$

$$\Leftrightarrow \quad x = \frac{V_0}{3}t + s$$



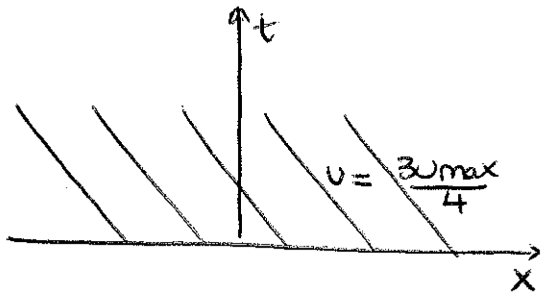
$\rightarrow$ since u is constant along a characteristic, we see that traffic is smoothly flowing at velocity $\frac{V_0}{3}$ and

$u(x,t) = \frac{U_{max}}{3}$ is always consta

Example 2 Now suppose $u(x,0) = \frac{3U_{max}}{4}$.

Then, following the same steps,

---> $x = -\frac{V_0}{2}t + s$



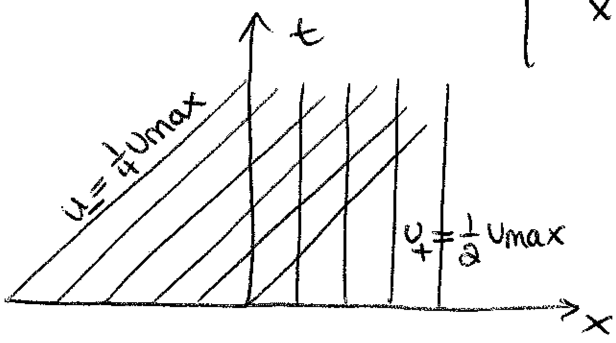
→ Again, the traffic density is always constant ($u(x,t) = \frac{3U_{max}}{4}$) and flowing at the velocity $V(u) = \frac{V_0}{4}$.

But as can be seen on the characteristics, information travels backwards at velocity $-\frac{V_0}{2}$ on the highway (more on this later).

Example 3 We now consider non-uniform ICs.

$$u(x,0) = \begin{cases} \frac{1}{4}U_{max} & \text{if } x<0 \\ \frac{1}{2}U_{max} & \text{if } x>0 \end{cases}$$

---> $$\begin{cases} x(s,\tau) = \frac{1}{2}V_0\tau + s & \text{if } s<0 \\ x(s,\tau) = s & \text{if } s>0 \end{cases}$$



→ A shock develops starting at $t=0$ at $x=0$.

Shock propagation: $\frac{d\gamma}{dt} = \frac{F(u_+) - F(u_-)}{u_+ - u_-} = ?$

$F(u) = uV(u)$

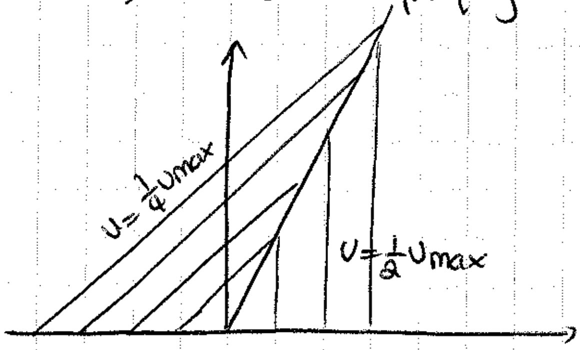
$= uV_0\left(1 - \frac{u}{U_{max}}\right)$

$$= \frac{\frac{1}{2}U_{max}V_0\left(\frac{1}{2}\right) - \frac{1}{4}U_{max}V_0\left(\frac{3}{4}\right)}{\frac{1}{2} - \frac{1}{4}}$$

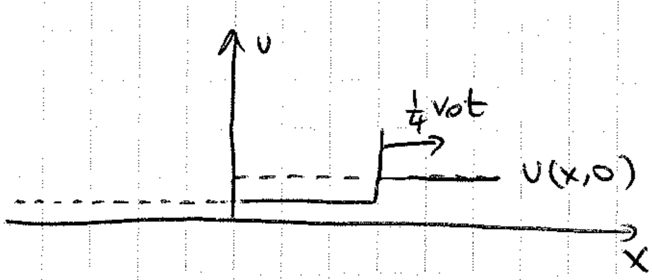
$$= \frac{1}{4}V_0$$

$\rightarrow \quad \gamma(t) = \frac{1}{4} v_0 t \qquad (\text{using } \gamma(0) = 0)$

$\Rightarrow$ The shock propagates at velocity $\frac{1}{4} v_0$.



$$u(x,t) = \begin{cases} \frac{1}{4} u_{max} & x < \frac{1}{4} v_0 t \\ \frac{1}{2} u_{max} & x > \frac{1}{4} v_0 t \end{cases}$$

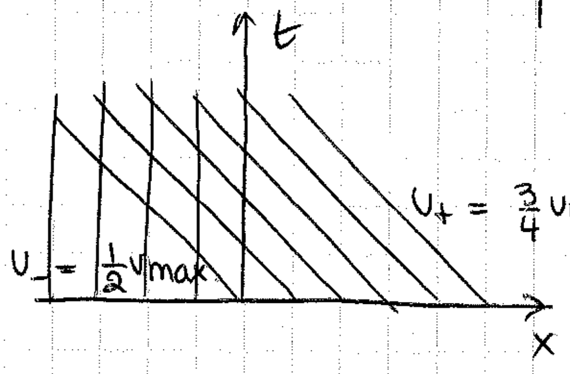


The shock disturbance propagates forward.

Note: Cars before the shock go at $\frac{3}{4} v_0$, while the shock goes at $\frac{1}{4} v_0 \Rightarrow$ all cars eventually meet the traffic disturbance, and they meet it at speed $\frac{1}{2} v_0$.

Example 4 Finally consider $u(x,0) = \begin{cases} \frac{1}{2} u_{max} & x < 0 \\ \frac{3}{4} u_{max} & x > 0 \end{cases}$

$$\dashrightarrow \begin{cases} x(s,\tau) = s & \text{if } s < 0 \\ x(s,\tau) = -\frac{v_0}{2} \tau + s & \text{if } s > 0 \end{cases}$$

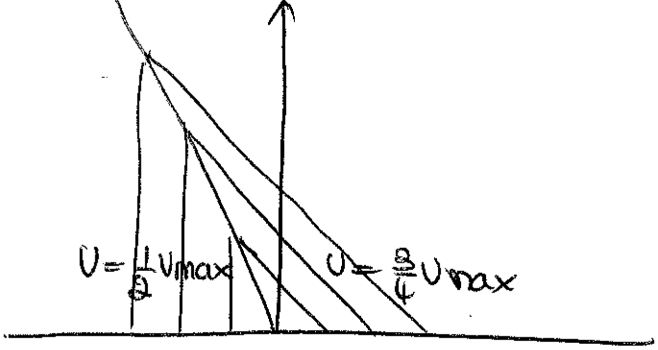


Following the same method, we find that

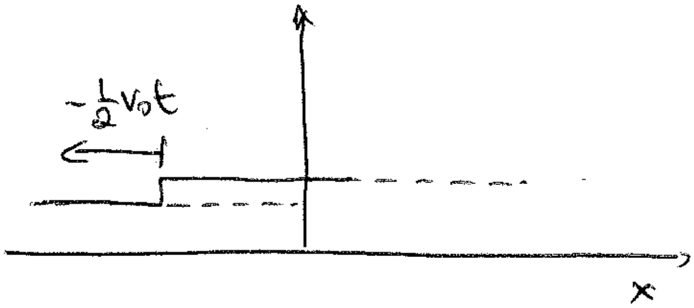
$$\frac{d\gamma}{dt} = \frac{F(u_+) - F(u_-)}{u_+ - u_-} = -\frac{1}{4} v_0$$

$$\Rightarrow \quad \gamma(t) = -\frac{1}{4} v_0 t$$

This time, the shock wave propagates backward on the highway...



$$u(x,t) = \begin{cases} \frac{1}{2} u_{max} & x < -\frac{1}{2} v_0 t \\ \frac{3}{4} u_{max} & x > -\frac{1}{2} v_0 t \end{cases}.$$



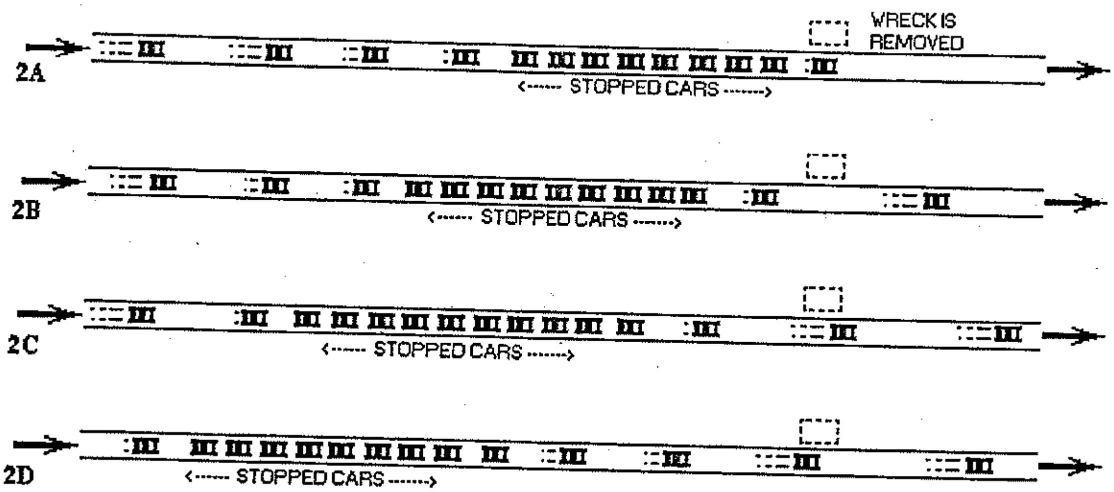
This time cars driving at velocity $\frac{v_0}{2}$ on highway encounter a disturbance travelling towards them at velocity $\frac{1}{4} v_0$

→ The relative velocity of cars & shock is high ($\frac{3}{4} v_0$) → high chance of accident.

<u>Applications.</u>

- on peak traffic hours, the speed limit on the London M25 is halved, in the hope of reducing the formation & propagation of traffic waves
- See movie.

Illustration of the cause of traffic waves



M25 data (London) of traffic density vs space & time