

Back to the first order PDE

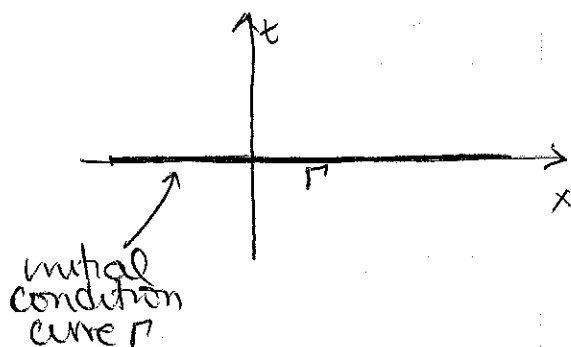
Step 1 - We represent the additional condition curve as a parametric curve with parameter s

Suppose we know $u(x, t)$ on a particular curve Γ in the (x, t) plane. Let's parametrize Γ with the functions $x_0(s), t_0(s)$ such that

$$\Gamma = \begin{cases} x_0(s) \\ t_0(s) \end{cases}$$

then on this curve $u(x_0(s), t_0(s)) = u_0(s)$

Examples: • Suppose we want to impose $u(x, 0) = 3x$



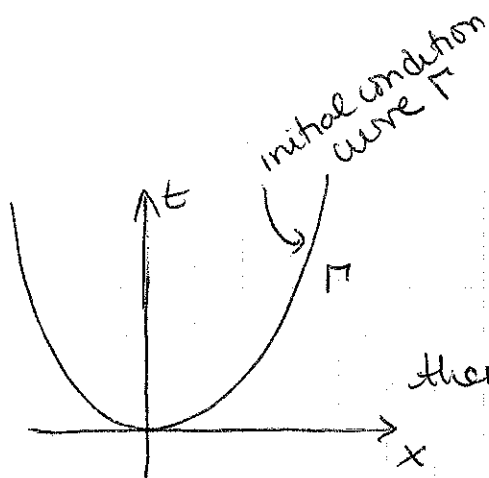
The initial condition curve has $t=0$ for all $x \rightarrow$

Parametrize it (for example) as

$$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \end{cases} \Rightarrow u_0(s) = 3s$$

or we could also use

$$\begin{cases} x_0(s) = s^2 \\ t_0(s) = 0 \end{cases} \Rightarrow u_0(s) = 3s^2$$



• Suppose $u(x, x^2) = e^x$

then

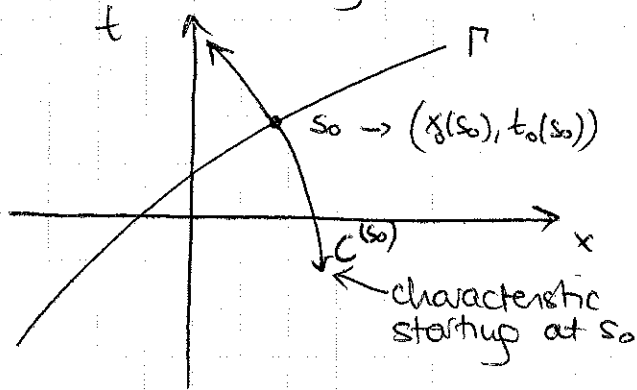
$$\begin{cases} x_0(s) = s \\ t_0(s) = s^2 \end{cases} \Rightarrow u_0(s) = e^s$$

Note: Since there are many possible parametric representations of the same curve, always try to choose the simplest one:

Prefer $\begin{cases} x_0(s) = s \\ t_0(s) = s^2 \end{cases}$ over $\begin{cases} x_0(s) = \ln(s^2) \\ t_0(s) = [\ln(s^2)]^2 \end{cases} !$

Step 2: Now that we have parametrized the "initial" condition curve we want to identify special curves along which the PDE behaves as an ODE. These are called characteristics.

- ① Suppose that for a selected point on the initial curve there exists only one characteristic emanating from it



⇒ let's parametrize this characteristic with the new parameter z

$$C^{(s_0)} = \begin{cases} x^{(s_0)}(z) \\ t^{(s_0)}(z) \end{cases}$$

⇒ on this curve

$$u(x^{(s_0)}(z), t^{(s_0)}(z)) = u^{(s_0)}(z)$$

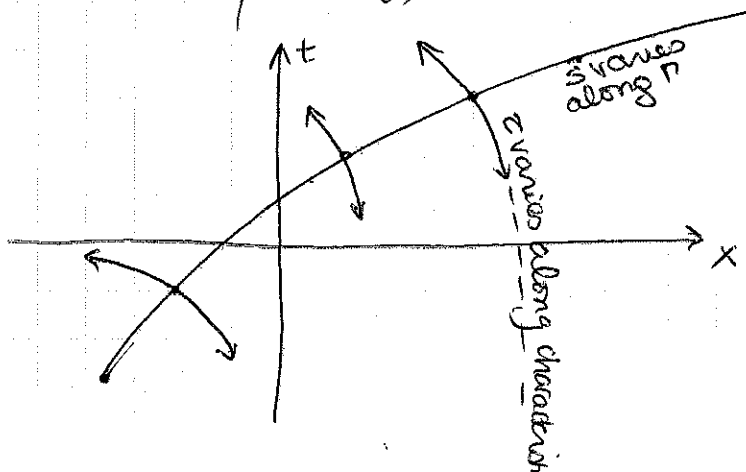
u depends on z only

- ② Now let's do this same construction for every point $[x_0(s), t_0(s)]$ on the initial condition curve

⇒ We get a family of characteristics, each starting from a point identified with the parameter s , and each parametrized with z :

$$C^{(s)} = \begin{cases} x^{(s)}(z) \\ t^{(s)}(z) \end{cases}$$

$$\text{with } u^{(s)}(z) = u(x^{(s)}(z), t^{(s)}(z))$$



⇒ Note that what we have really done, is to remap the (x, t) space onto the (s, z) space

so that the function

$$u(x, t) \text{ is also } u(x^{(s)}(z), t^{(s)}(z))$$

$$= u(s, z)$$

with the added requirement that the PDE

$$a \frac{\partial u}{\partial t} + b \frac{\partial u}{\partial x} \text{ behaves like an ODE}$$

when restricted to a characteristic ($s = \text{constant}$).

How do we impose this requirement?

Note that $\left. \frac{\partial u}{\partial z} \right|_s$ is the derivative of u along a characteristic.

(i.e. holding s constant)

↑
derivative of
 u w.r.t parameter z
at constant s

By multivariate chain rule, and using $\begin{pmatrix} x^{(s)}(z) \\ t^{(s)}(z) \end{pmatrix}$ on characteristic

$$\begin{aligned} \left. \frac{\partial u}{\partial z} \right|_s &= \frac{\partial u}{\partial x} \left. \frac{\partial x}{\partial z} \right|_s + \frac{\partial u}{\partial t} \left. \frac{\partial t}{\partial z} \right|_s \\ &= \frac{\partial u}{\partial x} \frac{d}{dz} [x^{(s)}] + \frac{\partial u}{\partial t} \frac{d}{dz} [t^{(s)}] \end{aligned}$$

Group back to the original PDE, if

$$a \frac{\partial u}{\partial t} + b \frac{\partial u}{\partial x} = \frac{d}{dz} [t^{(s)}] \frac{\partial u}{\partial t} + \frac{d}{dz} [x^{(s)}] \frac{\partial u}{\partial x}$$

$$\text{then } \Rightarrow \left. \frac{\partial u}{\partial z} \right|_s = c_1 u + c_0$$

Now the PDE looks like an ODE for s held constant, i.e., along a characteristic.

This occurs when

$$\boxed{\frac{dt^{(s)}}{dz} = a \quad \frac{dx^{(s)}}{dz} = b}$$

$$\Rightarrow \begin{cases} t^{(s)} = a\tau + \text{constant specific to this characteristic} \\ x^{(s)} = b\tau + \text{constant} \quad " \quad " \quad " \quad " \end{cases}$$

Suppose we require that when $\tau=0$ we are on the initial condition curve then at $\tau=0$

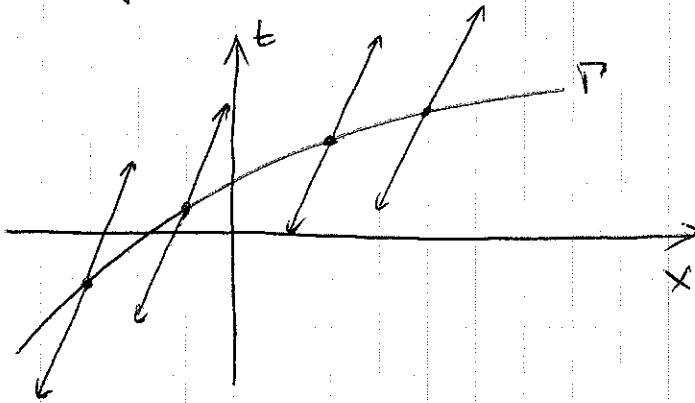
$$\begin{cases} t^{(s)} = t_0(s) \\ x^{(s)} = x_0(s) \end{cases}$$

thus $C = \int \begin{cases} t^{(s)} = a\tau + t_0(s) \\ x^{(s)} = b\tau + x_0(s) \end{cases}$

is the parametric equation for the characteristic emanating from the point $[x_0(s), t_0(s)]$ on the initial condition curve.

What do these characteristics look like?

Here, they are straight lines with slope $\frac{a}{b}$ in the (x, t) plane



Note: this is only true when the PDE has constant coefficients (see later)

Step 3 What is the solution to the PDE?

Now we have to solve $\frac{\partial u}{\partial \tau} = c_1 u + c_0$ for each s

$$\Rightarrow u = A(s) e^{c_1 \tau} - \frac{c_0}{c_1} + \frac{c_0}{c_1} e^{c_1 \tau} \quad (\text{check this})$$

where the arbitrary constant $A(s)$ is chosen such that $u = u_0(s)$ when $\tau = 0$:

$$u(s, \tau) = u_0(s) e^{c_1 \tau} - \frac{c_0}{c_1} + \frac{c_0}{c_1} e^{c_1 \tau}$$

How do we get a solution in terms of (x, t) ?

Invert the system (if possible)

$$\begin{cases} t = az + t_0(s) \\ x = b'z + x_0(s) \end{cases}$$

to write c and s in terms of x and t , then plug into $u(s, z)$

Example 1 Suppose we want to solve the simple transport equation

$$\begin{cases} \frac{\partial u}{\partial t} + v_0 \frac{\partial u}{\partial x} = 0 \\ u(x, 0) = e^{-x^2/2} \end{cases} \quad (\text{a Gaussian})$$

Step 1: Parametrize the initial condition

$$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = e^{-s^2/2} \end{cases}$$

Step 2 The characteristic curves are such that

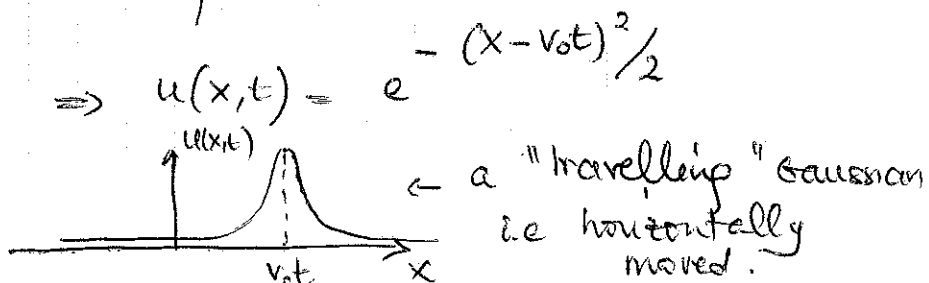
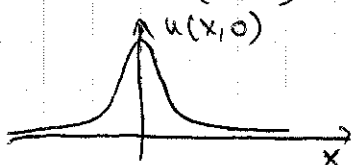
$$\begin{aligned} \left(\begin{array}{l} a=1 \\ b=v_0 \end{array} \right) & \begin{cases} \frac{\partial t^{(s)}}{\partial z} = 1 \\ \frac{\partial x^{(s)}}{\partial z} = v_0 \end{cases} \Rightarrow \begin{cases} t^{(s)} = z + t_0(s) \\ x^{(s)} = v_0 z + x_0(s) \end{cases} \Rightarrow \begin{cases} t^{(s)} = z \\ x^{(s)} = v_0 z + s \end{cases} \end{aligned}$$

Step 3: The solution to $\frac{\partial u}{\partial z} = 0$ ($C = C_0 = 0$)

is $u = \text{constant on a characteristic}$
 $\Rightarrow u = u_0(s) = e^{-s^2/2}$

Step 4: $\begin{cases} t = z \\ x = v_0 z + s \end{cases} \Rightarrow \begin{cases} z = t \\ s = x - v_0 t \end{cases}$

so $u(s, z) = e^{-s^2/2} \Rightarrow u(x, t) = e^{-(x - v_0 t)^2/2}$



← a "travelling" Gaussian i.e. horizontally moved.

Step 5 Always check your answer

$$\frac{\partial u}{\partial t} = -v_0 (x-v_0 t) e^{-\frac{(x-v_0 t)^2}{2}}$$

$$\frac{\partial u}{\partial x} = (x-v_0 t) e^{-\frac{(x-v_0 t)^2}{2}}$$

$$\frac{\partial u}{\partial t} + v_0 \frac{\partial u}{\partial x} = 0 \quad \checkmark$$

Example 2 : General case

$$\begin{cases} \frac{\partial u}{\partial t} + v_0 \frac{\partial u}{\partial x} = 0 \\ u(x, 0) = F(x) \end{cases}$$

Step 1

$$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = F(s) \end{cases}$$

Step 2

$$\begin{cases} \frac{dt}{dz} = 1 \\ \frac{dx}{dz} = v_0 \end{cases} \Rightarrow \begin{cases} t = z \\ x = v_0 z + s \end{cases}$$

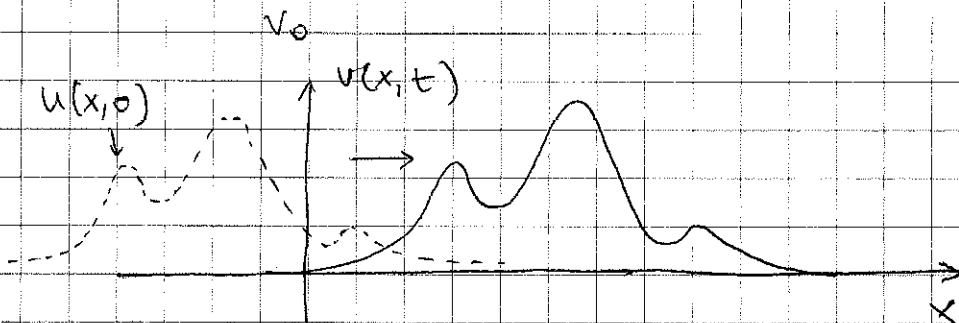
Step 3

$$\frac{du}{dz} = 0 \Rightarrow u = u_0(s) = F(s)$$

Step 4

$$\begin{cases} z = t \\ s = x - v_0 t \end{cases} \rightarrow \boxed{u(x, t) = F(x - v_0 t)}$$

\Rightarrow The initial condition $F(x)$ "moves" with velocity



Example 3:

$$\begin{cases} \frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = 0 \\ u(x, 0) = e^{-x^2} \end{cases}$$

Step 1:

$$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = e^{-s^2} \end{cases}$$

Step 2:

$$\begin{cases} \frac{dt}{dz} = 1 \\ \frac{dx}{dz} = x \end{cases} \Rightarrow \begin{cases} t = z + t_0(s) = z \\ x = x_0(s)e^z = se^z \end{cases}$$

Step 3:

$$\frac{du}{dz} = 0 \Rightarrow u = u_0(s) = e^{-s^2}$$

Step 4:

$$\begin{cases} z = t \\ s = xe^{-t} \end{cases} \Rightarrow u(x, t) = e^{-\frac{(xe^{-t})^2}{e^{2t}}} = e^{-\frac{x^2}{e^{2t}}}$$

This describes a Gaussian with constant amplitude, constant mean, but with a width which grows exponentially in time.

2.3.3 Semilinear equations

The method for semilinear equations:

$$\begin{cases} a(x, t) \frac{\partial u}{\partial t} + b(x, t) \frac{\partial u}{\partial x} = c(x, t, u) \\ u(x, 0) = \phi(x) \end{cases}$$

is the same as for linear equations. However, note that the resulting ODE for u will be nonlinear.

Example

$$\begin{cases} \frac{\partial u}{\partial t} + x \frac{\partial u}{\partial x} = -e^{-u} \\ u(x, 0) = e^{-x^2} \end{cases}$$

Step 1

$$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = e^{-s^2} \end{cases}$$

Step 2

$$\begin{cases} \frac{dt}{dz} = 1 \\ \frac{dx}{dz} = x \end{cases} \Rightarrow \begin{cases} t = z + t_0(s) = z \\ x = x_0(s)e^z = se^z \end{cases}$$

Step 3

$$\frac{du}{dz} = -e^{-u} \Rightarrow \frac{du}{e^{-u}} = -dz \Rightarrow e^u du = -dz$$
$$e^u = -z + k(s) \Rightarrow u = \ln(z + k(s))$$

At $z=0$ $e^{u_0(s)} = k(s) \Rightarrow k(s) = e^{e^{-s^2}}$

$$u = \ln(-z + e^{e^{-s^2}})$$

Step 4

$$\begin{cases} z = t \\ s = xe^{-t} \end{cases} \Rightarrow u(x, t) = \ln\left(-t + e^{e^{-x^2 e^{-2t}}}\right)$$

Step 5

Check:

$$\frac{\partial u}{\partial t} = \frac{-1 + \frac{2x^2 e^{-2t} e^{-x^2 e^{-2t}} e^{-x^2 e^{-2t}}}{e^{-x^2 e^{-2t}}}}{-t + e^{e^{-x^2 e^{-2t}}}}$$

$$\frac{\partial u}{\partial x} = \frac{-2xe^{-2t} e^{-x^2 e^{-2t}} e^{-x^2 e^{-2t}}}{-t + e^{e^{-x^2 e^{-2t}}}}$$

so ✓

What did we learn?

① Method of solution of ~~semi~~linear, first order PDEs

Step 1: Parametrize the initial condition curve

Step 2: If $a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t,u)$

then the characteristics are found by solving the system

$$\begin{cases} \frac{\partial t^{(s)}}{\partial z} = a(x,t) \\ \frac{\partial x^{(s)}}{\partial z} = b(x,t) \end{cases} \quad \left. \vphantom{\begin{cases} \frac{\partial t^{(s)}}{\partial z} = a(x,t) \\ \frac{\partial x^{(s)}}{\partial z} = b(x,t) \end{cases}} \right\} \text{Note that these are coupled ODEs.}$$

with the initial condition

$$t^{(s)}(z=0) = t_0(s)$$

$$x^{(s)}(z=0) = x_0(s)$$

Step 3: The solution to the PDE in (s, z) is found by solving

$$\frac{\partial u^{(s)}}{\partial z} = c(x, t, u)$$

(note that x and t depend on s and z)

Step 4: If possible, invert the system

$$\begin{cases} t(s, z) \\ x(s, z) \end{cases} \quad \text{to get} \quad \begin{cases} z(x, t) \\ s(x, t) \end{cases}$$

and plug into $u(s, z)$ to get $u(x, t)$.

Step 5 Check answer.

② Note:

- When the linear PDE is homogeneous ($c(x,t) = 0$) then

$$\frac{\partial u}{\partial z} = 0 \Rightarrow u \text{ is constant along characteristics; in other words, the characteristics are contour levels of the solution } u(x,t).$$

- When the PDE is not homogeneous then u is not constant along characteristics. The characteristics propagate the initial condition according to the equation

$$\frac{\partial u^{(s)}}{\partial z} = c(x^{(s)}(z), t^{(s)}(z), u^{(s)}(z))$$

(see examples later)

③ Question

- What is the condition for the mapping

$$\begin{cases} x(s, z) \\ t(s, z) \end{cases} \text{ to be invertible?}$$

- What happens if the characteristics are somewhere parallel to the initial condition curve?