Example of application

The gravitational potential of a galaxy made of stars of density $\rho(\underline{r})$ is the solution of

$$\nabla^2 \underline{\Phi} = 4\pi G \rho(\underline{r})$$

$$\Rightarrow \quad \Phi(\underline{r}) = \int_{\text{universe}} G(r, r') \, . \, 4\pi G \rho(\underline{r}') \, d^3\underline{r}'$$

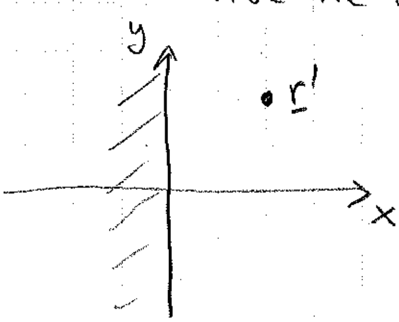
$$= \int_{\text{universe}} - \frac{\cancel{4\pi} G \rho(r') d^3\underline{r}'}{\cancel{4\pi} \, |r - r'|} = \underbrace{-\int \frac{G\rho(r') d^3 r'}{|r-r'|}}_{\text{should be familiar....}}$$

③ Greens functions in non-infinite domains

(a) The method of images for semi-infinite domains

This method, beyond the interesting "trick" used, is a good illustration of a more general principle we shall derive later.

Here we consider a domain D, say $\{x \geq 0\}$, with $y \in \mathbb{R}$.



→ What is the Green's function for the Laplacian, with homogeneous Dirichlet condition on $x=0$, with a δ at $\underline{r}'$?

We could try to solve directly for $\nabla^2 G = \delta(\underline{r} - \underline{r}')$ and $G = 0$ on $x = 0$ — However, this is actually quite tricky ... It's actually much easier to construct G by using the Green's funct° in the ∞ domain (see previous section.

Let Γ be the solution of $\nabla^2 \Gamma = \delta(r - \underline{r}')$ in the ∞ domain. We know

$$\Gamma(\underline{r}, \underline{r}') = \frac{1}{2\pi} \ln(|\underline{r} - \underline{r}'|)$$

By construction $\nabla^2 \Gamma$ is zero everywhere except at the singularity $\underline{r}'$

Now Γ cannot be G, because $\Gamma \neq 0$ on the $x=0$ axis.

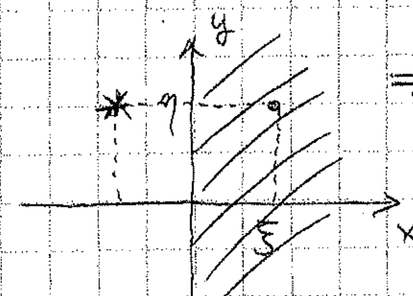
However, if we construct G as

$G = \Gamma + h$, then h must satisfy

$$\begin{cases} \nabla^2 h = 0 & \text{everywhere for } x>0 \\ h = -\Gamma & \text{on } x=0 . \end{cases}$$

Method of images (reflection principle)

Idea: Construct the function h to be the symmetric function of Γ across the domain boundary:



$\Rightarrow$ If Γ has a pole in (x', y'), construct a function that has a pole in $(-x', y')$

here $h = -\Gamma(x, y; -x', y')$

so that $G = \Gamma(x, y; x', y') - \Gamma(x, y; -x', y')$

We can verify that

$\nabla^2 G = \nabla^2 \Gamma = \delta(r - \underline{r}')$ (since $\nabla^2 h = 0$ everywhere in D (the pole is outside of D))

$G(x=0) = 0$

since $h(0, y, \xi, \eta) = -\Gamma(0, y; \xi, \eta)$

($\Gamma(x, y, x', y') = \frac{1}{2\pi} \ln \sqrt{(x-x')^2 + (y-y')^2}$ and $\Gamma(x, y, -x', y') = \frac{1}{2\pi} \ln \sqrt{(x+x')^2 + (y-y')^2}$

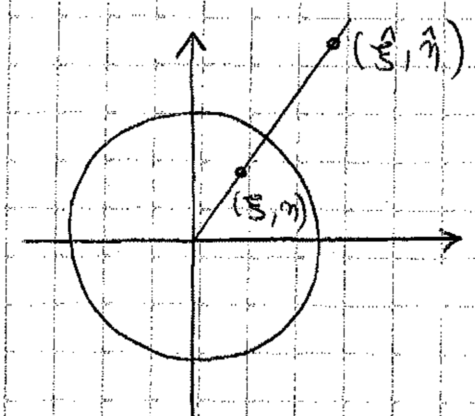
are equal at $x=0$)

(b) A "similar" construction can be used to compute h in a disk.

Consider the Dirichlet problem in the disk

$$\begin{cases} \nabla^2 \psi = f(x,y) & (x^2+y^2)^{1/2} < R \\ \psi(x,y) = g(x,y) & (x^2+y^2)^{1/2} = R \end{cases}$$

$\Rightarrow$ we want to find, for each (ξ,η), the function $h(x,y;\xi,\eta)$ such that



$$\begin{cases} \nabla^2 h = 0 & \text{in the disk} \\ h(x,y;\xi,\eta) = \Gamma(x,y;\xi,\eta) & \text{on the disk} \end{cases}$$

<u>Trick</u>: Consider the "inverse" point of (ξ,η), $(\hat{\xi},\hat{\eta})$ defined as

$$\begin{pmatrix} \hat{\xi} \\ \hat{\eta} \end{pmatrix} = \frac{R^2}{\xi^2+\eta^2}\begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

and let

$$h(x,y,\xi,\eta) = \Gamma\left[\frac{\sqrt{\xi^2+\eta^2}}{R}x, \frac{\sqrt{\xi^2+\eta^2}}{R}y; \frac{\sqrt{\xi^2+\eta^2}}{R}\hat{\xi}, \frac{\sqrt{\xi^2+\eta^2}}{R}\cdot\hat{\eta}\right]$$

$$= \Gamma\left[\frac{\sqrt{\xi^2+\eta^2}}{R}x, \frac{\sqrt{\xi^2+\eta^2}}{R}y; \frac{R}{\sqrt{\xi^2+\eta^2}}\xi, \frac{R}{\sqrt{\xi^2+\eta^2}}\eta\right]$$

Check:

(1) $\nabla^2 h = h_{xx} + h_{yy} = \frac{\xi^2+\eta^2}{R^2}\nabla^2\Gamma = 0$

(2) $h(x,y;\xi,\eta) = -\frac{1}{2\pi}\ln\left[\frac{\xi^2+\eta^2}{R^2}(x-\hat{\xi})^2 + \frac{\xi^2+\eta^2}{R^2}(y-\hat{\eta})^2\right]$

should be equal to

$\Gamma(x,y;\xi,\eta) = -\frac{1}{2\pi}\ln\left[(x-\xi)^2+(y-\eta)^2\right]$

on the circle $x^2+y^2=R^2$

$$\frac{\xi^2+\eta^2}{R^2}\left[(x-\hat{\xi})^2+(y-\hat{\eta})^2\right]$$
$$= \frac{\xi^2+\eta^2}{R^2}\left[x^2+y^2 - 2x\hat{\xi} - 2y\hat{\eta} + \hat{\xi}^2+\hat{\eta}^2\right]$$
$$= (\xi^2+\eta^2) - (2x\xi + 2y\eta) + \frac{R^2}{\xi^2+\eta^2}(\xi^2+\eta^2)$$
$$= R^2 - 2x\xi + 2y\eta + (\xi^2+\eta^2)$$
$$= (x-\xi)^2 + (y-\eta)^2 \qquad \square.$$

$\Rightarrow$ The Green's function on the disk of radius R is

$$G(x,y;\xi,\eta) = \Gamma(x,y;\xi,\eta) - \Gamma\left(\frac{\sqrt{\xi^2+\eta^2}}{R}x, \frac{\sqrt{\xi^2+\eta^2}}{R}y; \frac{R}{\sqrt{\xi^2+\eta^2}}\xi, \frac{R}{\sqrt{\xi^2+\eta^2}}\right.$$
$$= -\frac{1}{2\pi}\ln\left((x-\xi)^2+(y-\eta)^2\right)$$
$$+ \frac{1}{2\pi}\ln\left[\left(\frac{\sqrt{\xi^2+\eta^2}}{R}x - \frac{R}{\sqrt{\xi^2+\eta^2}}\xi\right)^2 + \left(\frac{\sqrt{\xi^2+\eta^2}}{R}y - \frac{R}{\sqrt{\xi^2+\eta^2}}\eta\right)^2\right]$$

and the solution to the Dirichlet problem

$$\begin{cases}\nabla^2 u = f & \text{in } D\\ u = g & \text{on } D\end{cases} \quad \text{is} \quad u(\xi,\eta) = -\iint_D G f\,dxdy - \int_{\partial D} g\frac{\partial G}{\partial r}\,d\ell$$

(r = radial coordinate

④ General Considerations

The idea of constructing a Green's function in a bounded domain using the infinite domain one is quite general, e.g.

Given a domain D, we have

$$\begin{cases} \nabla^2 G = \delta(\underline{r}-\underline{r}') \\ G = 0 \text{ on } \partial D \end{cases}$$

$$\Leftrightarrow \begin{cases} G = \Gamma + h \quad \text{where} \\ \nabla^2 \Gamma = \delta(r-\underline{r}') \qquad \nabla^2 h = 0 \\ h = -\Gamma \text{ on } \partial D. \end{cases}$$

This provides a way of transforming the problem of finding the solution to a δ-function forcing with homogeneous conditions, to a Laplace problem with non-homogeneous conditions

This is not often easy however. See "tricks" learned earlier.

For generalizations to non-homogeneous BCs, and Von-Neuman bcs, see Pinchover & Rubinstein textbook

⑤ Solutions by coordinate change.

Solutions in domains with more complicated geometries can often be found by finding a mapping which transforms the domain into a basic one (disk, lines, etc). Conformal mappings are very powerful for this.

CHAPTER 2   First order PDEs (in 2 dimensions)

2.1 General formulae

A first order PDE in 2 dimensions is in the form of

$$F\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) = 0$$

- A first order linear PDE in 2 dimensions is

$$a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t)u + d(x,t).$$

NONLINEAR PDES:

- A first order semilinear PDE in 2D is

$$a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t,u)$$

- A first order quasilinear PDE is

$$a(x,t,u)\frac{\partial u}{\partial t} + b(x,t,u)\frac{\partial u}{\partial x} = c(x,t,u).$$

A fully nonlinear first order PDE is none of the above!

2.2 Method of characteristics for quasilinear equations

2.2.1 Warmup example

Let's study $\quad u_t = c_0 u + c_1(x,t) \quad$ c_0 constant

Note that for each x, it is actually an ODE in t

→ fix x and solve it!

Use integrating factor method (for example)

$$u_t - c_0 u = c_1(x,t)$$

⇒ We try to find an integrating factor $\mu(x,t)$ such that

$$\mu u_t - \mu C_0 u = \mu G(x,t)$$

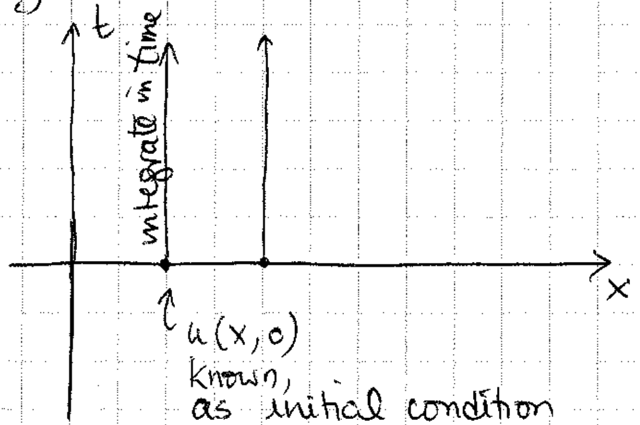
$$= \frac{\partial}{\partial t}(\mu u)$$

$\longrightarrow$ take $\mu = e^{-C_0 t}$ so

$$\frac{\partial}{\partial t}\left(e^{-C_0 t} u\right) = e^{-C_0 t} G(x,t)$$

$$e^{-C_0 t} u(x,t) - e^{-C_0 \cdot 0} u(x,0) = \int_{t'=0}^{t'=t} e^{-C_0 t'} G(x,t')\,dt'$$

Again, this can be done for each value of x separately: we are solving the equation by integrating along lines of constant x



<u>Initial conditions (u is known at $t=0$)</u>

Suppose we require that $u(x,0) = 3x$ then

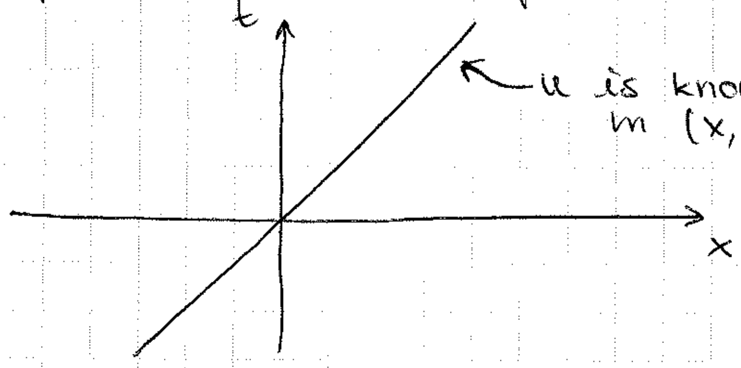
$$u(x,t) = e^{+C_0 t} u(x,0) + \int_{t'=0}^{t'=t} e^{-C_0(t'-t)} G(x,t')\,dt'$$

$$= 3x e^{+C_0 t} + \int_{t'=0}^{t'=t} e^{-C_0(t'-t)} G(x,t')\,dt'$$

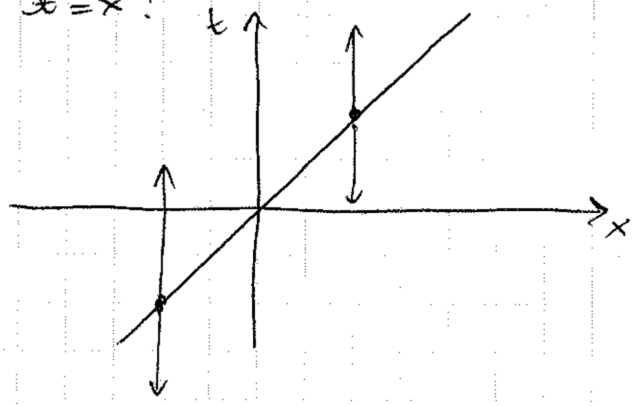
$\rightarrow$ a unique solution.

Other kinds of additional condition

① Suppose instead we require that $u(x,x) = 3x$



Then, instead of integrating from $t'=0$, we integrate from $t'=x$:



Mathematically:

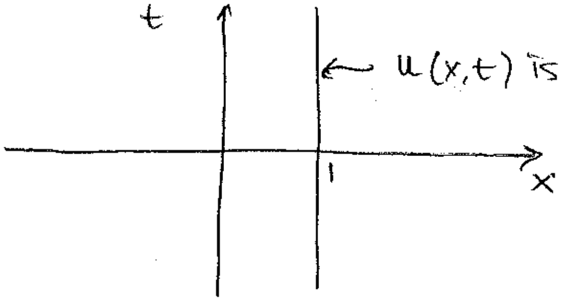
$$e^{-c_0 t} u(x,t) - e^{-c_0 x} u(x,x) = \int_{t'=x}^{t'=t} e^{-c_0 t'} G(x,t')\,dt'$$

$$\Rightarrow \quad e^{-c_0 t} u(x,t) = e^{-c_0 x} \cdot 3x + \int_x^t e^{-c_0 t'} G(x,t')\,dt'$$

$$u(x,t) = e^{-c_0(x-t)} \cdot 3x + \int_x^t e^{-c_0(t'-t)} G(x,t')\,dt'$$

→ again, there is a unique solution to the PDE with the given additional condition.

② Now suppose we set $G = 0$ and try to impose as additional condition $u(1,t) = 2t$



<u>Problem!</u> The additional condition doesn't satisfy the equation

$$\frac{\partial u}{\partial t} = 2 \quad \rightarrow \quad u_t - C_0 u = 2 - 2C_0 t \neq 0 .$$

→ there are no solutions to the equation!

③ Now suppose $u(1,t) = 2e^{C_0 t}$ then

$$u_t - C_0 u = 2C_0 e^{C_0 t} - 2C_0 e^{C_0 t} = 0 \quad ✓$$

⟹ the additional condition satisfies the equation

But note that any function of the form
$$u(x,t) = f(x)\, e^{C_0 t}$$
satisfies the PDE <u>and</u> the additional condition provided $f(1) = 2$

⟹ there are an ∞ of solutions to the problem!

<u>Conclusion</u>: • Depending on the additional conditions chosen, there can be one, no or an ∞ of solutions to the problem. Case ① is <u>well-posed</u> while cases ② and ③ are <u>ill-posed</u>

• What is the difference between cases ①, ② and ③? Note that in case ①, the additional condition crosses all lines of constant x, while in cases ② and ③, the additional condition <u>is</u> a line of constant x.

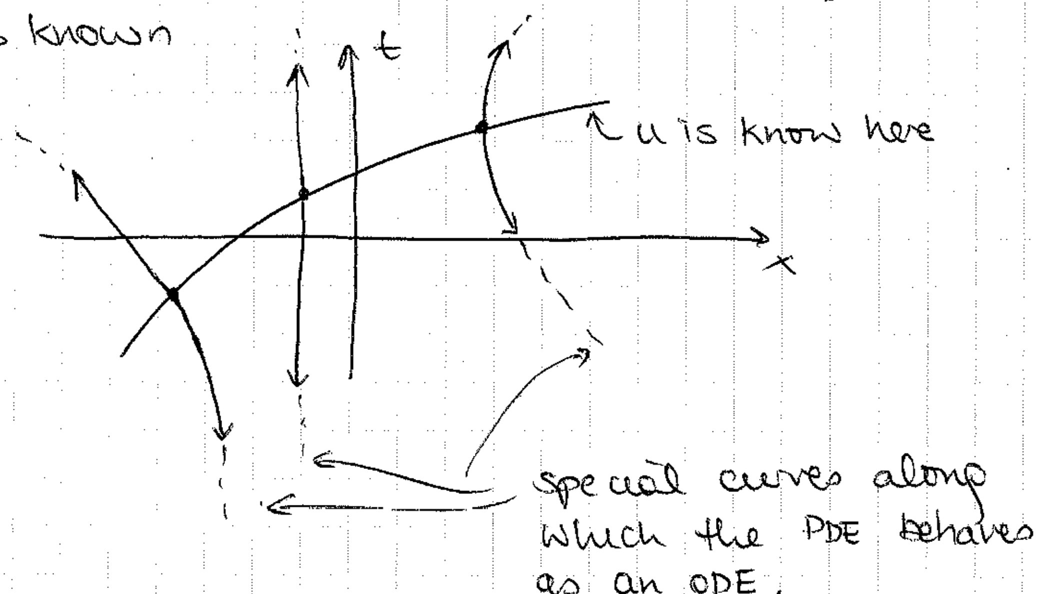
2.2.2 Gearing up to the general method

Now consider the linear transport equation with constant coefficients.

$$a \frac{\partial u}{\partial t} + b \frac{\partial u}{\partial x} = c_1 u + c_0$$

where a, b, c_1, c_0 are constants.

__Idea:__ we would like to find curves (as before) along which we could integrate the PDE __as if__ it were an ODE, from an initial or additional condition line where $u(x,t)$ is known



DETOUR: __Review of parametric curves__

Any curve in $\mathbb{R}^n$ can be represented by a set of parametric equations

$$\begin{cases} x_1 = f_1(s) \\ x_2 = f_2(s) \\ \vdots \\ x_n = f_n(s) \end{cases}$$

where s is the parameter.

__Examples:__ A circle in $\mathbb{R}^2$ (x,y) centred on $(0,0)$ has the equation $\begin{cases} x = R\cos(s) \\ y = R\sin(s) \end{cases}$ where R is the radius

- A straight line in $\mathbb{R}^2$ has the parametric equation

$$\begin{cases} x = as + c \\ y = bs + d \end{cases}$$

Check: eliminate s to get

$$y = b\left(\frac{x-c}{a}\right) + d = \frac{b}{a}x + \left(d - \frac{bc}{a}\right)$$

Property of parametric curves

The tangent vector to the curve $\{f_1(s) \dots f_n(s)\}$ is

$$\underline{df} = \begin{pmatrix} df_1/ds \\ df_2/ds \\ \vdots \\ df_n/ds \end{pmatrix}$$

Examples: • the tangent vector to the line

$$\begin{cases} x = as + c \\ y = bs + d \end{cases} \quad \text{is} \quad \underline{df} = \begin{pmatrix} dx/ds \\ dy/ds \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

• Suppose you are travelling from SC to Big Sur. Your trajectory is given by the parametric curve

$$\begin{pmatrix} x(t) \\ y(t) \\ h(t) \end{pmatrix} \begin{matrix} \leftarrow \text{latitudinal position } x \\ \leftarrow \text{longitudinal position } y \\ \leftarrow \text{height} \end{matrix}$$

Your velocity is the tangent vector to the trajectory

$$\underline{v} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dh}{dt} \end{pmatrix} \begin{matrix} \leftarrow \text{North-South velocity} \\ \leftarrow \text{East-West velocity} \\ \leftarrow \text{vertical velocity.} \end{matrix}$$

Note: A parametrization is NOT unique.

Example: $\begin{cases} x = R\sin s \\ y = R\cos s \end{cases}$ and $\begin{cases} x = R\sin(s^2) \\ y = R\cos(s^2) \end{cases}$ represent the same curve