(c) <u>Variation of constants (variation of parameters)</u>

For completeness note that there is another method for solving forced linear ODEs, which however (to my knowledge) does not carry over in higher dimensions.

see RHB 15.2.4 for detail (our textbook is not great for that).

Given $\mathcal{L}(u) = f(x)$

- The general solution of the homogeneous equation $\mathcal{L}(u)=0$ is

$$u_G(x) = \alpha_1 u_1(x) + \alpha_2 u_2(x)$$

- We seek a solution of the forced equation as

$$u_p(x) = \beta_1(x)\, u_1(x) + \beta_2(x)\, u_2(x)$$

i.e. we "vary" the constants.

where $\beta_1(x)$ and $\beta_2(x)$ are TBD.

- The original equation, $\mathcal{L}(u_p) = f(x)$, provides <u>one</u> constraint on the 2 functions β_1 and β_2. We are free to select a second constraint; we do so in a way which simplifies the algebra required in solving for β_1 and β_2.

<u>Trick</u>: Require that $\boxed{\beta_1'(x)\, u_1(x) + \beta_2'(x)\, u_1(x) = 0}$.

(see RHB for generalizations to higher-order ODEs).

Why does this help?

Note: ' notation and $_x$ notation used interchangably here.

$$\mathcal{L}(u_p) = f(x) \Rightarrow \left(p(x)\, u_{px}\right)_x + q(x)\, u_p = f(x)$$

$$\Rightarrow \Big[p(x) \left[\beta_1 u_1 + \beta_2 u_2 \right]_x \Big]_x + q(x) \left[\beta_1 u_1 + \beta_2 u_2 \right] = f(x)$$

$$\Rightarrow \Big[p(x) \Big\{ \underbrace{\cancel{\beta_1' u_1 + \beta_2' u_2}}_{0 \text{ by other constraint}} + \beta_1 u_{1x} + \beta_2 u_{2x} \Big\} \Big]_x + q\left(\beta_1 u_1 + \beta_2 u_2\right) = f(x)$$

$$\Rightarrow \quad \underbrace{\beta_1 (p u_{1x})_x + \beta_2 (p u_{2x})_x + q\beta_1 u_1 + q\beta_2 u_2}_{\substack{=0 \text{ since} \\ u_1,\, u_2 \text{ are solutions} \\ \text{of } \mathcal{L}(u)=0.}} + \beta_{1x}\, p u_{1x} + \beta_{2x}\, p u_{2x} = f(x)$$

$$\Rightarrow \quad \boxed{\beta_1' u_1' + \beta_2' u_2' = \frac{f(x)}{p(x)}}$$

So now we have 2 equations for two functions β_1' and β_2', which can be integrated to get $\beta_1(x)$ and $\beta_2(x)$.

<u>Example</u> $\quad \dfrac{d^2u}{dx^2} + \omega^2 u = f(x)$. with $u(0)=0$ $u(L)=0$.

→ The two general homogeneous solutions are $\cos\omega x$ and $\sin\omega x$ → we seek

$$u_p(x) = \beta_1(x)\cos\omega x + \beta_2(x)\sin\omega x.$$

→ We know that

$$\begin{cases} \beta_1'\cos\omega x + \beta_2'\sin\omega x = 0 \\ -\beta_1'\omega\sin\omega x + \omega\beta_2'\cos\omega x = f(x) \end{cases}$$

$$\Rightarrow \begin{cases} \beta_1' = -\dfrac{f(x)\sin\omega x}{\omega} \\ \beta_2' = \dfrac{f(x)}{\omega}\cos\omega x \end{cases}$$

$$\Rightarrow \text{ So } \quad \beta_1(x) = -\int_0^x \frac{f(x')\sin\omega x'}{\omega}\,dx' + \beta_1(0)$$

$$\beta_2(x) = +\int_0^x \frac{f(x')\cos\omega x'}{\omega}\,dx' + \beta_2(0)$$

To fit the BCs we want $u(0)=0$ and $u(L)=0$

$$u(0) = 0 = \beta_1(0)\,u_1(0) + \beta_2(0)u_2(0)$$
$$= \beta_1(0) \Rightarrow \beta_1(0) = 0$$

$$u(L) = 0 = \beta_1(L)\,u_1(L) + \beta_2(L)\,u_2(L)$$
$$= -\int_0^L \frac{f(x')\sin\omega x'}{\omega}\,dx' \cdot \cos\omega L$$
$$+ \left[\int_0^L \frac{f(x')\cos\omega x'}{\omega}\,dx' + \beta_2(0)\right]\sin\omega L$$
$$\rightarrow \quad \beta_2(0) = -\int_0^L \frac{f(x')\cos\omega x'}{\omega}\,dx'.$$
$$+ \int_0^L \frac{f(x')\sin\omega x'}{\omega\tan\omega L}\,dx'$$

$$\Rightarrow \beta_1(x) = -\int_0^x f(x')\,\frac{\sin\omega x'}{\omega}\,dx'$$

$$\beta_2(x) = -\int_x^L \frac{f(x')\cos\omega x'}{\omega}\,dx' + \int_0^L \frac{f(x')\sin\omega x'\,dx'}{\omega\tan\omega L}.$$

$$\Rightarrow u(x) = -\cos\omega x\int_0^x f(x')\,\frac{\sin\omega x'}{\omega}\,dx'$$
$$+ \sin\omega x\left\{-\int_x^L \frac{f(x')\cos\omega x'}{\omega}\,dx' + \int_0^L \frac{f(x')\sin\omega x'\,dx'}{\omega\tan\omega L}\right\}$$

$$= \int_0^L G(x,x')\,dx' \quad \text{provided}$$

$$G(x,x') = \begin{cases} -\dfrac{\cos\omega x\sin\omega x'}{\omega} + \dfrac{\sin\omega x'\sin\omega x}{\omega\tan\omega L} & x' < x \\[2ex] -\dfrac{\sin\omega x\cos\omega x'}{\omega} + \dfrac{\sin\omega x'\sin\omega x}{\omega\tan\omega L} & x' > x \end{cases}$$

<u>as before</u>

II Green's functions in higher dimensions; the Poisson equation

In what follows, we now move to the problem of Green's functions in 2 or more dimensions, with specific application to the Poisson equation.

$$\begin{cases} \nabla^2 u = f(\underline{r}) & r \in D \\ \text{Some boundary condition} & r \in \partial D . \end{cases}$$

As we shall see, for this particular kind of problem the existence and uniqueness of solutions is not guaranteed.

1 Existence & uniqueness of solutions

The problem with the Poisson equation comes from the following property:

$$\int_V \nabla \cdot u \, dV = \int_{\partial V} \underline{u} \cdot \underline{n} \, dS \qquad \text{the divergence theorem}$$

$\underline{n}$ = unit vector normal to surface (outward).

Since, by definition, $\nabla^2 u = \nabla \cdot (\nabla u)$ then the integral of the Poisson equation over the domain D is

$$\int_D \nabla \cdot (\nabla u) \, dV = \underbrace{\int_{\partial V} \nabla \underline{u} \cdot \underline{n} \, dS}_{\substack{\text{depends on boundary conditions} \\ \text{(sometimes known)}}} = \underbrace{\int_D f(\underline{r}) \, dV}_{\substack{\text{depends on forcing} \\ \text{(known).}}}$$

$\Rightarrow$ The nature of the equation itself provides a constraint on the "flux" of $\underline{u}$ through the boundary and the integrated source term.

The mathematical constraint can be interpreted physically once more by thinking of the Poisson equation as the steady-state version of a heat diffusion process $\Rightarrow$

in equilibrium the heat flux through the boundary of a domain must be equal to the integrated heat sources in the domain.

However, we see here that this is a very general constraint.

!! In particular, it implies that the Poisson equation with Neumann-type boundary conditions (i.e when

$\underline{\nabla u} \cdot \underline{n}$ is known on the boundary)

does NOT have a solution unless $\int_{\partial D} \nabla u \cdot n \, ds = \int_{D} f(r) \, dV$.

In general: • the problem of existence of solutions for elliptic equations is much more complex than for parabolic / hyperbolic equations.

• Provided the domain considered is bounded and smooth enough.

- Solutions to the Dirichlet problem exist and are unique
- solutions to the Neumann problem exist (if $*$ holds) but are not unique ($u = v + K$, $K \in \mathbb{R}$ is also solution)
- solutions to the Robin problem exist and are unique.

See Pinchover & Rubinstein textbook, for good proofs + much more.

As in 1D, the solution to $\nabla^2 u = f(\underline{r})$ with homogeneous boundary conditions can be written as

$$u(\underline{r}) = \int_D G(\underline{r},\underline{r}')\, f(\underline{r}')\, d^n\underline{r}'$$

↑ Volume integral in n dimensions

where the Green's function G is the solution of

$$\nabla^2 G(\underline{r},\underline{r}') = \delta(\underline{r}-\underline{r}').$$

with the same boundary conditions.

The problem of "finding" G, however, is often quite difficult.

Of course if separation of variables can be done, then the Green's function can be constructed by eigenfunction expansion (see previous chapters for examples). In other cases, additional methods exist.

② The Green's functions for the unbounded domain

The solution to $\nabla^2 G(\underline{r},\underline{r}') = \delta(\underline{r}-\underline{r}')$ in the infinite domain is easily found from the solution of

$$\nabla^2 G(\underline{r}) = \delta(\underline{r}) \quad \text{by translation symmetry.}$$

Meanwhile, the solution to $\nabla^2 G = \delta(\underline{r})$ is point symmetric, and so becomes an equation in r only when expressed in a polar coordinate system (in 2D) or a spherical coordinate system (in 3D).

Example in 2D : the infinite domain Green's function is s.t.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) = \delta(r)$$

$\rightarrow$ this is really $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) = 0$ everywhere but $r = 0$

$\Rightarrow$ we seek a solution singular at $r = 0$:

$$\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) = 0$$

$$\Rightarrow \quad r\frac{\partial G}{\partial r} = K \quad \Rightarrow \quad \frac{\partial G}{\partial r} = \frac{K}{r}$$

$$\Rightarrow \quad G(r) = K' + K\ln r$$

- Typically, BCs in an ∞ domain require $G \to 0$ as $r \to \infty$. Here of course, it's a bit of a problem... Let's take $K' = 0$. (sweep under carpet).
- The remaining constant comes from the requirement that the integral of a δ-function over any domain be one $\Rightarrow$ on a "unit circle" for example

$$\int_{\text{unit circle}} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) \cdot r\,dr\,d\theta = \int_{\text{unit circle}} \delta(r)\, r\,dr\,d\theta = 1$$

$$\Rightarrow \quad \int_0^1 \frac{\partial}{\partial r}\left(r\frac{\partial G}{\partial r}\right) 2\pi\, dr = 1$$

$$\Rightarrow \quad 2\pi\left[r\frac{\partial G}{\partial r}\right]_0^1 = 1$$

$$\Rightarrow \quad \left.\frac{\partial G}{\partial r}\right|_{r=1} = \frac{1}{2\pi} \quad \Rightarrow \quad K = \frac{1}{2\pi}$$

$$\Rightarrow \quad \boxed{K = \frac{1}{2\pi}}$$

$$\Rightarrow \quad G(r) = \frac{1}{2\pi}\ln r$$

$$\Rightarrow \quad \boxed{G(r, r') = \frac{1}{2\pi}\ln\left(|r - r'|\right)}$$

Example of application :

The steady state temperature distribution on an infinite conducting plate subject to heating $H(x,y)$ is the solution of

$$\nabla^2 T = -H(x,y)$$

and so
$$T(x,y) = \int_{\mathbb{R}^2} -H(x',y')\,G(x,x';y,y')\,dx'dy'$$

$$T(x,y) = \iint_{\mathbb{R}^2} -H(x',y')\ \ln\left((x-x')^2+(y-y')^2\right)\frac{dx'dy'}{4\pi}.$$

whatever the function H is.

Example in 3D. The infinite domain Green's function is

solution of $\quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right) = \delta(r)$, i.e, the singular

solution of $\quad \frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right) = 0.$

$$\Rightarrow \quad r^2\frac{\partial G}{\partial r} = K \quad \Rightarrow \quad \frac{\partial G}{\partial r} = \frac{K}{r^2}$$

$$\Rightarrow \quad G = K' - \frac{K}{r}.$$

Again, if we require $G \to 0$ as $r \to \infty$, we can take $K' = 0$

The integral of $\delta(\underline{r})$ over a unit sphere must be one hence

$$\int_{\substack{\text{unit}\\ \text{sphere}}} \cancel{\frac{1}{r^2}}\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right)\cancel{r^2}\,dr\,\sin\theta\,d\theta\,d\phi = 1$$

$$\Rightarrow \quad 4\pi\left[r^2\frac{\partial G}{\partial r}\right]_0^1 = 1 \quad \Rightarrow \quad K = \frac{1}{4\pi}$$

So $\quad \boxed{G(r) = \frac{-1}{4\pi r}} \quad \Rightarrow \quad G(r,r') = \frac{-1}{4\pi|r-r'|}$