Hyperbolic case : similarly

$$\sum_n \ddot{a}_n(t) v_n(x) + \sum \lambda_n a_n(t) v_n(x) = \sum b_n(t) v_n(x)$$

so by orthogonality

$$\ddot{a}_n(t) + \lambda_n a_n(t) = b_n(t)$$

This time we use the Laplace transform method:

$$s^2 \hat{a}_n - s a_n(0) - a_n'(0) + \lambda_n \hat{a}_n = \hat{b}_n(s)$$

$$\Rightarrow \quad \hat{a}_n(s) = \frac{\hat{b}_n(s) + s a_n(0) + a_n'(0)}{s^2 + \lambda_n}$$

Now, the λ_n are positive $\Rightarrow$ the inverse Laplace transform (see Maple) is

$$a_n(t) = \frac{1}{\sqrt{\lambda_n}} \int_0^t b_n(t') \sin\left(\sqrt{\lambda_n}(t-t')\right) dt'$$

$$+ a_n(0) \cos\left(\sqrt{\lambda_n} t\right) + a_n'(0) \frac{1}{\sqrt{\lambda_n}} \sin\left(\sqrt{\lambda_n} t\right)$$

$\Rightarrow$ The general solution of the problem becomes

$$u(x,t) = \sum_{n=0}^{\infty} v_n(x) \cdot \left[\frac{1}{\sqrt{\lambda_n}} \int_0^t b_n(t') \sin\left(\sqrt{\lambda_n}(t-t')\right) dt' + a_n(0) \cos\left(\sqrt{\lambda_n} t\right) + a_n'(0) \frac{1}{\sqrt{\lambda_n}} \sin\left(\sqrt{\lambda_n} t\right) \right]$$

but with $b_n(t') = \int_a^b \frac{F(x',t') v_n(x') r(x') dx'}{\|v_n^2\|}$

we get

$$u(x,t) = \sum \left[a_n(0) \cos\left(\sqrt{\lambda_n} t\right) + a_n'(0) \frac{1}{\sqrt{\lambda_n}} \sin\left(\sqrt{\lambda_n} t\right) \right] \cdot v_n(x)$$

$$+ \int_0^t \int_a^b F(x',t') G(x,x';t,t') dx' dt'$$

with

$$G(x,x';t,t') = \sum_{n=0}^{\infty}{}' \frac{1}{\sqrt{\lambda_n}} \sin\left(\sqrt{\lambda_n}(t-t')\right) \frac{v_n(x)v_n(x')r(x')}{\|v_n^2\|}$$

↳ the wave kernel

Example of the bridge

Recall: $u_{tt} - c^2 u_{xx} = \sin\left(\frac{2\pi x}{L}\right)\cos(\omega t)$

$u = 0$ at both ends

$u_t(x,0) = u(x,0) = 0$

$\Rightarrow$ Eigenmodes/values of spatial homogeneous pb:

$$\begin{cases} v_n(x) = \sin\left(\frac{n\pi x}{L}\right) \\ \lambda_n = \frac{n^2\pi^2c^2}{L^2} \end{cases}$$

then $u(x,t) = \sum{}' \left[a_n(0)\cos\left(\frac{n\pi ct}{L}\right) + a_n'(0)\frac{L}{n\pi c}\sin\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$

$$+ \int_0^t\int_0^L F(x',t')G(x,x';t,t')dx'dt'$$

Fitting this to ICs $\Rightarrow$ $a_n(0) = a_n'(0) = 0$

$$u(x,t) = \int_0^t dt' \int_0^L dx' \sin\left(\frac{2\pi x'}{L}\right)\cos(\omega t')\sum_{n=0}^{\infty}{}' \frac{L}{n\pi c}\sin\left(\frac{n\pi c}{L}(t-t')\right)\frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\frac{L}{2}}$$

$$= \int_0^t dt' \cos(\omega t')\frac{L}{2\pi c}\sin\left(\frac{2\pi c}{L}(t-t')\right)\sin\left(\frac{2\pi x}{L}\right)$$

$$= \int_0^t \frac{dt'}{2}\left[\sin\left(\omega t' + \frac{2\pi c}{L}(t-t')\right) - \sin\left(\omega t' - \frac{2\pi c}{L}(t-t')\right)\right]\sin\frac{2\pi x}{L}\frac{L}{2\pi c}$$

$$= \frac{1/2}{\omega + \frac{2\pi c}{L}}\left[\cos\left(\frac{2\pi ct}{L}\right) - \cos\omega t\right] - \frac{1/2}{\omega - \frac{2\pi c}{L}}\left[\cos\left(\frac{2\pi c}{L}\right) - \cos\omega t\right]\frac{L}{2\pi c} \cdot \sin\left(\frac{2\pi x}{L}\right)$$

$$= \frac{1}{\omega^2 - 4\pi^2c^2/L^2}\left[\cos\left(\frac{2\pi ct}{L}\right) - \cos\omega t\right]\sin\left(\frac{2\pi x}{L}\right) \checkmark$$

CHAPTER 8: Green's functions

In this chapter we generalize the notion of a Green's function. We will focus here on time-independent problems, for simplicity. We will show that the solution to non-homogeneous problems can often be written as the convolution of the "forcing" with a Green's function, the latter being unique to each PDE & boundary conditions.

We start with the 1D problem to illustrate properties of Greens functions & learn how to construct them.

(I) Green's functions in 1D

Let's consider examples of the kind

$$\mathcal{L}(u) = f(x) \quad \text{with} \quad \mathcal{L}(u) = (p(x)u_x)_x + q(x)u$$

and homogeneous boundary conditions

$$\begin{cases} \alpha u(a) + \beta u'(a) = 0 \\ \delta u(b) + \gamma u'(b) = 0 \end{cases}$$ (see 9.3.5 for non homogeneous bcs)

We will discuss 2 different ways of finding the function $G(x,x')$ such that the solution to the problem, $u(x)$, is

$$u(x) = \int_a^b G(x,x') f(x') \, dx'$$

(a) Eigenvalue/eigenfunction expansion

As before, let's consider the SL problem

$$(p(x)u_x)_x + q(x)u = -\lambda u \quad \text{with the same BCs}$$

→ this has an ∞ sequence of eigenvalues λ_n with corresponding eigenvectors $v_n(x)$

We know that we can write any function $u(x)$ satisfying the BCs as:

$$u(x) = \sum_n{}' \alpha_n v_n(x)$$

$\Rightarrow$ Plugging into the equation, we get

$$\mathcal{L}\left(\sum_n{}' \alpha_n v_n(x)\right) = \sum_n{}' \alpha_n \mathcal{L}(v_n(x)) = \sum_n{}' -\lambda_n \alpha_n v_n(x) = f(x)$$

By orthogonality,

$$\alpha_n = -\frac{\langle f(x), v_n(x)\rangle}{\lambda_n \langle v_n, v_n\rangle} = -\frac{\int_a^b f(x') v_n(x') dx'}{\lambda_n \int_a^b v_n^2(x') dx'}$$

(provided $\lambda_n \neq 0$)

So finally,

$$u(x) = \sum_n{}' - v_n(x) \frac{\int_a^b f(x') v_n(x') dx'}{\lambda_n \int_a^b v_n^2(x') dx'}$$

$$= \int_a^b f(x') G(x, x') dx'$$

provided $\quad G(x, x') = \sum_n{}' - \frac{v_n(x) v_n(x')}{\lambda_n \|v_n\|^2}$

$\Rightarrow$ This is one possible method for constructing the Green's function.

Example: What is the general solution to

$$\frac{d^2u}{dx^2} + \omega^2 u = f(x) \qquad u(0)=0 \quad u(L)=0 \;?$$

- Eigenfunctions: $\dfrac{d^2u}{dx^2} + \omega^2 u = -\lambda u$

$$\Rightarrow \quad v_n(x) = \left\{ \begin{matrix} \cos\left(\sqrt{\lambda+\omega^2}\, x\right) \\ \sin\left(\sqrt{\lambda+\omega^2}\, x\right) \end{matrix} \right\}$$

By bcs, we have to have $\sqrt{\lambda + \omega^2} = \frac{n\pi}{L}$

so $\quad \lambda_n = \frac{n^2\pi^2}{L^2} - \omega^2.$

so $\quad v_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

$\Rightarrow$ as long as we are off resonnance ($\lambda_n \neq 0$), we can write

$$G(x,x') = \sum_n - \frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\left(\frac{n^2\pi^2}{L^2}-\omega^2\right)\frac{L}{2}}$$

$$= -\frac{2}{L}\sum_n{}' \frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\frac{n^2\pi^2}{L^2}-\omega^2}.$$

The solutions to any possible forcing can be found by convolving G with the forcing... But what does G look like? (see maple file).

(b) Solution of the δ-function forcing.

Another interesting, and much more fundamental method comes from noticing that G is the solution of

$$(*) \quad \mathcal{L}(G(x,x')) = \delta(x-x') \quad \text{with the same bcs.}$$

Indeed, remember that $\mathcal{L}$ is a linear operator, so

$$\mathcal{L}\left(\int_a^b G(x,x')f(x')dx'\right)$$

$$= \int_a^b \mathcal{L}(G(x,x'))f(x')dx') \quad \leftarrow \text{since } \mathcal{L} \text{ operates on the variable } x, \text{ not } x'$$

$$= \int_a^b \delta(x-x')f(x')dx' \quad \leftarrow \text{if indeed } G \text{ is solution of } (*)$$

$$= f(x) \quad \left(\text{Note that the } \delta\text{-function is an even function so } \delta(x-x') = \delta(x'-x)\right)$$

$\Rightarrow$ The problem is now shifted to: how to find solutions of

$$\mathcal{L}(G) = \delta(x-x').$$

In order to do this, note that $\delta(x-x')$ is effectively 0 unless $x=x'$ so we mostly solve

$$\mathcal{L}(G)=0 \quad \text{except at } x=x'.$$

Since $\mathcal{L}$ is a 2nd order linear operator, the general solution which can be written as a linear combination of 2 basic functions.

Hence $\mathcal{L}(G)=0$

$$\Rightarrow \quad G(x,x') = \alpha u_1(x) + \beta u_2(x)$$

There will be one such solution on either side of the point x' $\Rightarrow$

On the left $(x<x')$ $\quad G_L(x,x') = \alpha_L u_1(x) + \beta_L u_2(x)$

On the right $(x>x')$ $\quad G_R(x,x') = \alpha_R u_1(x) + \beta_R u_2(x)$

Two of these 4 unknown coefficients can be found by fitting the boundary conditions, at $x=a$ and $x=b$.

The other 2 are found by requiring that

- $G(x,x')$ be <u>continuous</u> at $x=x'$
- The derivatives of $G(x,x')$ on either side of $x=x'$ satisfy the correct "jump" condition, obtainde by integrating. (*) across $x=x'$:

$$\int_{x'-\epsilon}^{x'+\epsilon} \mathcal{L}(G)\,dx = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x-x')\,dx = 1$$

$$\int_{x'-\epsilon}^{x'+\epsilon} \left[(p(x)G_x)_x + q(x)G\right] dx = 1$$

$$\lim_{\epsilon\to 0}\left\{\left[\, p(x'+\epsilon)\,G_x\big|_{x'+\epsilon} - p(x'-\epsilon)\,G_x\big|_{x'-\epsilon}\right\} = 1$$

(since q, u continuous)

$\hookrightarrow \lim_{\epsilon \to 0} \left[\frac{dG}{dx}\Big|_{x=x'+\epsilon} - \frac{dG}{dx}\Big|_{x=x'-\epsilon} \right] = \frac{1}{p(x')}$

$$\hookrightarrow \quad \alpha_R u_1'(x') + \beta_R u_2'(x') - \alpha_L u_1'(x) - \beta_L u_2'(x') = \frac{1}{p(x')}$$

(The fourth condition).

While this method is very general, it is more easily understood when applied to a particular example.

<u>Example</u>: $\frac{d^2u}{dx^2} + \omega^2 u = f(x)$. (Same as before)

We now seek $G(x,x')$, the solution of

$$\frac{d^2G}{dx^2} + \omega^2 G = \delta(x-x')$$

- The equation $\frac{d^2G}{dx^2} + \omega^2 G = 0$ has a general solutions of the kind $G = \alpha \cos\omega x + \beta \sin\omega x$.

$\Rightarrow$ let's consider the 2 solutions on either side of x':

On left: $G_L(x) = \alpha_L \cos\omega x + \beta_L \sin\omega x$

From $u(0)=0$ we get $G(0,x')=0$ so $G_L(0) = 0 \Rightarrow \alpha_L = 0$ so $G_L(x) = \beta_L \sin\omega x$

On right: $G_R(x) = \alpha_R \cos\omega x + \beta_R \sin\omega x$

From $u(L)=0$ we get $G(L,x')=0$ so $G_R(L)=0 \Rightarrow$

$$\alpha_R \cos(\omega L) + \beta_R \sin(\omega L) = 0$$

$$\Rightarrow \frac{\alpha_R}{\beta_R} = -\tan(\omega L) \Rightarrow \alpha_R = -\beta_R \tan(\omega L)$$

- From continuity at $x=x'$ we get

$$\beta_L \sin(\omega x') = -\beta_R \tan(\omega L)\cos(\omega x') + \beta_R \sin(\omega x')$$

So $\quad \beta_L = \beta_R \left(1 - \frac{\tan\omega L}{\tan\omega x'}\right)$

- Finally, the derivative jump condition implies

$$\left.\frac{dG_R}{dx}\right|_{x'} - \left.\frac{dG_L}{dx}\right|_{x'} = 1 \qquad (\text{here}, p(x) = 1)$$

$$\Rightarrow \quad \beta_R \omega \tan\omega L \sin\omega x' + \beta_R \omega \cos\omega x' - \beta_L \omega \cos\omega x' = 1$$

So

$$\beta_R \left[\omega \tan\omega L \sin\omega x' + \omega\cos\omega x' - \omega\cos\omega x'\left(1 - \frac{\tan\omega L}{\tan\omega x'}\right)\right] = 1$$

$$\beta_R = \frac{1}{\omega \tan(\omega L)} \frac{1}{\sin\omega x' + \frac{\cos\omega x'}{\tan\omega x'}}$$

$$= \frac{\sin(\omega x')}{\omega \tan(\omega L)}$$

$$\beta_L = \frac{1}{\omega \tan(\omega L)} \left(\sin\omega x' - \tan\omega L \cos\omega x'\right)$$

$$= \frac{1}{\omega}\left(\frac{\sin\omega x'}{\tan\omega L} - \cos\omega x'\right)$$

$$\alpha_R = -\frac{\sin(\omega x')}{\omega}$$

$$\Rightarrow \quad G(x,x') = \begin{cases} \frac{1}{\omega}\left(\frac{\sin\omega x'}{\tan\omega L} - \cos\omega x'\right)\sin\omega x & \text{if } x < x' \\ -\frac{\sin\omega x'}{\omega}\cos\omega x + \frac{\sin\omega x' \sin\omega x}{\omega \tan\omega L} & x > x' \end{cases}$$

It can be shown (most easily by plotting them, see maple) that this is exactly the same function as before.