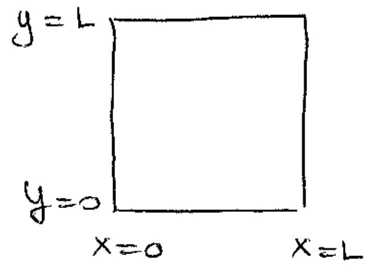
③ <u>Poisson equation</u>

Suppose we want to solve $\nabla^2 T = -H(x,y)$

$\uparrow$ heating source

to obtain the steady-state temperature profile in a metallic plate, heated as prescribed by $H(x,y)$ and with $T=0$ on all 4 sides; take $k=1$




Note that the $-$ sign comes from

$$\frac{\partial T}{\partial t} = \nabla^2 T + H(x,y)$$

$\rightarrow$ in steady state $\nabla^2 T = -H$.

The spatial eigenmodes in x-direction are (see previous lectures), for $T(0,y) = T(L,y) = 0$

$$A_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

$\rightarrow$ Assume $T(x,y) = \sum_{n=1}^{\infty} A_n(x) B_n(y)$

then

$$\sum_{n=1}^{\infty} -\frac{n^2\pi^2}{L^2} A_n(x) B_n(y) + A_n(x) \frac{d^2 B_n}{dy^2} = -H(x,y)$$

Noting that $\int_0^L A_n(x) A_m(x)\, dx = \frac{L}{2} \delta_{mn}$,

$$\frac{d^2 B_n}{dy^2} - \frac{n^2\pi^2}{L^2} B_n = -\frac{2}{L} \int_0^L H(x,y) \sin\left(\frac{n\pi x}{L}\right) dx.$$

$$= -h_n(y).$$

Suppose that $H(x,y) = \delta\left(x-\frac{L}{2}\right)\delta\left(y-\frac{L}{2}\right)$ to model a point source

Then $$\frac{d^2 B_n}{dy^2} - \frac{n^2\pi^2}{L^2} B_n = -\sin\left(\frac{n\pi}{2}\right) \delta\left(y - \frac{L}{2}\right) \cdot \frac{2}{L}$$

## Laplace transforms, review

Laplace transforms are very useful for solving non-homogeneous linear ODEs.

Idea: let $f$ be a function of $t$

The Laplace transform of $f$ is

$$\mathcal{L}(f) = \hat{f}(p) = \int_0^\infty f(t) e^{-pt}\, dt.$$

Properties: $\mathcal{L}(f') = p\hat{f}(p) - f(0)$

since $$\int_0^\infty \frac{df}{dt} e^{-pt}\, dt = \left[ f e^{-pt} \right]_0^\infty + \int_0^\infty p f e^{-pt} dt$$
$$= p\hat{f}(p) - f(0)$$

$$\mathcal{L}(f'') = p^2 \hat{f}(p) - p f(0) - f'(0)$$
(proof is similar).

So given a linear ODE with constant coefficients

$$af'' + bf' + cf = g(t) \qquad (*)$$

$$\mathcal{L}(*) \Rightarrow \quad a\left[p^2 \hat{f}(p) - pf(0) - f'(0)\right]$$
$$+ \quad b\left[p\hat{f}(p) - f(0)\right]$$
$$+ \quad c\hat{f}(p) = \int_0^\infty g(t) e^{-pt}\, dt = G(p)$$

Suppose $f(0)$ and $f'(0)$ are known (initial value problem) then this is an algebraic equation for $\hat{f}(p)$.

To recover $f(t)$, we need to do an inverse Laplace transform.

For detail on Inverse Laplace transforms, see handout. Usually, it's easy to find the solution using Inverse Laplace transform tables.

<u>Here</u>:

$$\frac{d^2B_n}{dy^2} - \frac{n^2\pi^2}{L^2} B_n = -\sin\left(\frac{n\pi}{2}\right) \delta\left(y-\frac{L}{2}\right) \cdot \frac{2}{L}$$

$$\Rightarrow \quad p^2 \hat{B}_n - p B_n(0) - B_n'(0) - \frac{n^2\pi^2}{L^2} \hat{B}_n = -\frac{2}{L} \sin\left(\frac{n\pi}{2}\right) \int_0^\infty \delta\left(y-\frac{L}{2}\right) e^{-py}\, dy$$

$$= -\sin\left(\frac{n\pi}{2}\right) e^{-p\frac{L}{2}} \cdot \frac{2}{L}$$

$B_n(0) = 0$ but $B_n'(0)$ is unknown. Let's leave it as is for the moment.

$$\Rightarrow \quad \hat{B}_n \left[ p^2 - \frac{n^2\pi^2}{L^2} \right] = B_n'(0) - \frac{2}{L} \sin\left(\frac{n\pi}{2}\right) e^{-p\frac{L}{2}}$$

so
$$\hat{B}_n(p) = \frac{B_n'(0)}{p^2 - \frac{n^2\pi^2}{L^2}} - \frac{2}{L} \sin\left(\frac{n\pi}{2}\right) e^{-p\frac{L}{2}} \frac{1}{p^2 - \frac{n^2\pi^2}{L^2}}$$

From tables:

- The inverse transform of $\frac{1}{p^2-a^2}$ is $\frac{\sinh(ay)}{a}$
- The inverse transform of

$$\frac{e^{-pb}}{p^2-a^2} \text{ is } \begin{cases} \frac{\sinh(a(y-b))}{a} & \text{if } y > b \\ 0 & \text{if } 0 < y < b \end{cases}$$

$$\Rightarrow \quad B_n(y) = \frac{B_n'(0)}{\frac{n\pi}{L}} \cdot \sinh\left[\frac{n\pi y}{L}\right] - \frac{2}{L}\sin\left(\frac{n\pi}{2}\right)\sinh\left(\frac{n\pi(y-\frac{L}{2})}{L}\right)\frac{1}{\frac{n\pi}{L}} \quad \text{if } y > \frac{L}{2}$$

At $y = L$, the solution is such that $B_n(L) = 0 \Rightarrow$

$$B_n'(0)\sinh(n\pi) - \frac{2}{L}\sin\left(\frac{n\pi}{2}\right)\sinh\left(\frac{n\pi}{2}\right) = 0$$

$$\Rightarrow \quad B_n'(0) = \frac{2}{L}\frac{\sin\left(\frac{n\pi}{2}\right)\sinh\left(\frac{n\pi}{2}\right)}{\sinh(n\pi)}.$$

So finally, we have

$$T(x,y) = \sum_{n=1}^{\infty}{}' \sin\left(\frac{n\pi x}{L}\right)\frac{\sin\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{L}}\frac{2}{L}\left[\frac{\sinh\left(\frac{n\pi}{2}\right)}{\sinh(n\pi)}\sinh\left(\frac{n\pi y}{L}\right) - \sinh\left(\frac{n\pi(y-\frac{L}{2})}{L}\right)\underbrace{H\left(y-\frac{L}{2}\right)}_{\text{Heaviside function}}\right]$$

<u>Note</u>: This expression is slightly awkward, but it can be shown that it indeed leads to the correct behavior in $y$, which should be symmetry across the $y = \frac{L}{2}$ line. (Homework!)

## II: General: Non-homogeneous equations; introduction to Green's functions

### (skip) 1 Non homogeneous (regular) SL problems (ODEs)

Given the ODE $\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = F(x)$

with bcs $\begin{cases} \alpha u(a) + \beta u'(a) = 0 \\ \gamma u(b) + \delta u'(b) = 0 \end{cases}$

① Seek solutions of the homogeneous eigenvalue eq.

$$\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = -\lambda u$$

→ this yields the eigenfunctions $\{v_n\}$ and eigenvalues $\{\lambda_n\}$

② Write $F(x) = \sum_n b_n v_n(x)$

(with $b_n = \int_a^b r(x')F(x')v_n(x')\,dx'$, if the $v_n$s are properly normalized)

Then since we know that the solution can also be written as

$$u(x) = \sum_n a_n v_n(x)$$

we can write

$$\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = \sum_n -\lambda_n a_n v_n(x) = \sum_n b_n v_n(x)$$

and by identification, $a_n = -\frac{b_n}{\lambda_n}$

$$\Rightarrow \quad u(x) = \sum_n -\frac{b_n}{\lambda_n} v_n(x) = -\sum_n \int_a^b \frac{r(x')}{\lambda_n} F(x')v_n(x')v_n(x)\,dx'$$

$$= \int_a^b G(x;x')F(x')\,dx'$$

where $\quad G(x;x') = -\sum_n \frac{1}{\lambda_n} v_n(x') v_n(x)\, r(x')$

- $G(x;x')$ is called the Green's function of the S.L problem

- It only depends on the characteristics of the homogeneous problem ($\{v_n\}$, $\{\lambda_n\}$) but, when integrated through with the forcing term $F(x)$, yields the solution of the forced problem

- Note that if the $\{v_n\}$ are not normalized then

$$G(x;x') = -\sum_n \frac{1}{\|v_n\|^2} \frac{r(x')}{\lambda_n} v_n(x') v_n(x) .$$

$$\text{where } \|v_n\|^2 = \int_a^b r(x) v_n(x)^2 \, dx .$$

<u>Example</u> Consider

$$y'' + y = 3\sin(2\pi x) \qquad \begin{matrix} y(0) = 0 \\ y(1) = 0 \end{matrix}$$

We seek the eigenfunctions of $y'' + y = -\lambda y$

$$\rightarrow \quad y'' + (1+\lambda) y = 0$$

$$\text{so} \quad y = \alpha \cos(\sqrt{1+\lambda}\, x) + \beta \sin(\sqrt{1+\lambda}\, x)$$

$$\text{with} \begin{cases} \alpha = 0 \\ \sqrt{1+\lambda_n} = n\pi \end{cases}$$

$$\Rightarrow \begin{cases} \lambda_n = n^2\pi^2 - 1 \\ v_n(x) = \sin(n\pi x) \end{cases} .$$

$\Rightarrow$ The Green's function $G(x,x') = -\sum_n \frac{\sin(n\pi x)\sin(n\pi x')}{\lambda_n \|\sin n\pi x\|^2}$

$$= -\sum_n \frac{2}{n^2\pi^2 - 1} \sin(n\pi x) \sin(n\pi x')$$

so the solution to the problem is $y(x) = \int_0^1 G(x,x')F(x')dx'$

$$y(x) = \int_0^1 \sum_n \frac{3\sin(2\pi x')}{1-n^2\pi^2} 2\sin(n\pi x)\sin(n\pi x')\,dx'$$

$$= \frac{3}{1-4\pi^2}\sin(2\pi x)$$

## 2. Application to parabolic/hyperbolic PDEs.

Now consider either $u_t - \frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = F(x,t)$

or $u_{tt} - \frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = F(x,t)$.

Idea: Solve the associated Sturm-Liouville problem of the homogeneous PDE

$$\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] + \lambda u = 0$$

to find the eigenvalues and eigenfunctions

$$\{v_n\},\ \{\lambda_n\}$$

then expand

$$F(x,t) = \sum_n b_n(t)v_n(x)$$

(in this case, $b_n(t) = \int_a^b F(x,t)r(x)v_n(x)\,dx$)

Assume a solution of the form

$$u(x,t) = \sum_n a_n(t)v_n(x)$$

and try the ansatz into the equation:

Parabolic case:
$$\sum_n \dot{a}_n(t)v_n(x) - \frac{1}{r(x)}\left[\left(p(x)\sum_n a_n(t)v_n'(x)\right)' + q(x)\sum a_n(t)v_n(x)\right]$$
$$= \sum b_n(t)v_n(x)$$

so that

$$\sum_n{}' \dot{a}_n(t) v_n(x) + \sum_n{}' \lambda_n a_n(t) v_n(x) = \sum_n b_n(t) v_n(x)$$

and (by orthogonality):

$$\dot{a}_n + \lambda_n a_n = b_n(t)$$

↳ integrating factor method:

$$\frac{d}{dt}\left(a_n e^{\lambda_n t}\right) = b_n(t) e^{\lambda_n t}$$

so $$a_n(t) e^{\lambda_n t} - a_n(0) = \int_0^t b_n(t') e^{\lambda_n t'} dt'$$

$$\Rightarrow a_n(t) = a_n(0) e^{-\lambda_n t} + e^{-\lambda_n t} \int_0^t b_n(t') e^{\lambda_n t'} dt'.$$

Putting it all together we find that

$$u(x,t) = \sum_n{}' a_n(0) e^{-\lambda_n t} v_n(x) + \int_0^t \sum_{n'}{}' e^{-\lambda_n (t-t')} v_n(x)\, b_n(t')\, dt'$$

$$= \sum_n{}' a_n(0) e^{-\lambda_n t} v_n(x) + \int_0^t \int_a^b \sum_{n'}{}' e^{-\lambda_n (t-t')} v_n(x) v_n(x') r(x')\, F(x',t')\, dx' dt'$$

So we can write

$$u(x,t) = \sum_n a_n(0) e^{-\lambda_n t} v_n(x) + \int_0^t \int_a^b G(x,t;x',t') F(x',t')\, dx' dt'$$

$$\text{with } G(x,t;x',t') = \sum_n{}' e^{-\lambda_n (t-t')} v_n(x) v_n(x') r(x')$$

Here G is called the Heat Equation Kernel.
↳ another example of a Green's function.

↳ u is the sum of
- the solution to the problem with no forcing
- \+ the weighted integral of $F(x,t)$ with the Green's function.

Example of the drunks exiting the pub.

Recall:
$$\frac{\partial p}{\partial t} = k\frac{\partial^2 p}{\partial x^2} + S(x,t)$$
$$\begin{cases} p(x,0)=0 \\ \frac{\partial p}{\partial x} = 0 \text{ at } x=0,L \\ S(x,t) = S_0 e^{-\frac{t}{\tau}}\,\delta\left(x-\frac{L}{2}\right) \quad \text{for } t>0. \end{cases}$$
(take $t_0=0$).

Homogeneous problem; separation of variables to get spatial eigenmodes $\Rightarrow$
$$\begin{cases} V_n(x) = \cos\left(\frac{n\pi x}{L}\right) \\ \lambda_n = \frac{n^2\pi^2}{L^2} \end{cases}$$

So, by the previous calculation, we have
$$u(x,t) = \sum_{n=0}^{\infty}{}' a_n(0)\, e^{-\lambda_n t} V_n(x) + \int_0^t\int_0^L S(x',t')\,G(x,x';t,t')\,dx'dt'$$

where $a_n(0)$ are obtained by fitting $u$ to initial conditions
$$u(x,0) = \sum_{n=0}^{\infty}{}' a_n(0) V_n(x) = 0 \Rightarrow a_n(0)=0$$

and where
$$G(x,x';t,t') = \sum_{n=0}^{\infty} e^{-\lambda_n(t-t')}\frac{V_n(x')\,V_n(x)\,r(x)}{\|V_n^2\|}$$
but $r(x)=1$

$$\Rightarrow u(x,t) = \int_0^t\int_0^L\sum_{n=0}^{\infty} S_0 e^{-\frac{t'}{\tau}}\,\delta\left(x'-\frac{L}{2}\right) e^{-\frac{n^2\pi^2}{L^2}(t-t')}\frac{1}{\|V_n^2\|}\cos\left(\frac{n\pi x'}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx'dt'$$
$$= \int_0^t\sum_{n=0}^{\infty} S_0 e^{-\frac{t'}{\tau}} e^{-\frac{n^2\pi^2}{L^2}(t-t')}\cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{L}\right)dt'\cdot\frac{1}{\|V_n^2\|}$$
$$= \sum_{n=0}^{\infty} S_0\cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{L}\right)\frac{1}{\frac{n^2\pi^2}{L^2}-\frac{1}{\tau}}\left[e^{-\frac{t}{\tau}} - e^{-\frac{n^2\pi^2}{L^2}t}\right]\frac{1}{\|V_n^2\|}$$