# II Partial Differential equations (PDEs)

## ① Mathematical definition

- A PDE is a functional relation between a function and its partial derivatives :

let $f : \mathbb{R}^n \to \mathbb{R}$ then the most general form of a PDE is

$$\underbrace{\mathcal{F}}_{\text{a functional of}} \Big( \underbrace{x_i}_{\substack{\text{the}\\\text{independent}\\\text{variables}}} ; f ; \underbrace{\frac{\partial f}{\partial x_i}}_{\substack{\text{the first}\\\text{order partial}\\\text{derivatives}}} ; \underbrace{\frac{\partial^2 f}{\partial x_i \partial x_j}}_{\substack{\text{the second}\\\text{order partial}\\\text{derivatives, etc...}}} ; \dots \Big) = 0 \qquad \begin{cases} i = 1 \dots n \\ j = 1 \dots n \end{cases}$$

- The order of the PDE is the order of the highest derivative involved
- A PDE is homogeneous if $f = 0$ is a solution
- A PDE is said to be linear if $\mathcal{F}$ is a linear combination of $f$ and its derivatives :

$$a_0(x_1,\dots,x_n) f + a_1(x_1 \dots x_n) \frac{\partial f}{\partial x_1} + \dots + a_n(x_1 \dots x_n) \frac{\partial f}{\partial x_n}$$
$$+ a_{11}(x_1 \dots x_n) \frac{\partial^2 f}{\partial x_1^2} + \dots \quad a_{nn}(x_1 \dots x_n) \frac{\partial^2 f}{\partial x_n^2} + \dots$$
$$= b(x_1 \dots x_n)$$

- A linear PDE is homogeneous if $b = 0$
- A non linear PDE is not linear

Examples : $f_y + f_x = 0$ is linear, homogeneous, first-order

$f_{xx} + x f_y = 0$ is linear, second order, homogeneous

$3x f_x + f_{yy} = 4$ is linear, second order, non-homogeneous

$f_y + f_x f_{yy} = 0$ is non-linear.

- There exists systems of PDEs describing the coupled evolution of two or more dependent variables

Example : 
$$\begin{cases} f_x + g_y f_{xx} = 0 \\ g_{xx} = 3 f_y \end{cases}$$
(a nonlinear coupled system of PDEs)

## ② PDEs in real systems

- Nature is classically represented by 3 spatial dimensions and one time dimension

  ⇒ Most PDEs studied involve these 4 variables, or a subset of them

- The study of PDEs is often an attempt at modelling Nature

  ⇒ There are usually 3 steps to the problem
  1. to construct a PDEs which describes the problem (cf Applied Mathematics)
  2. to solve the PDE mathematically
  3. to analyse the answer in the light of the problem

- This class mostly focusses on ② although there are two aspects of ① worth describing here

  - notion of covariance
  - selection of a coordinate system

## a. Notion of covariance

- The idea of a coordinate system is dependent on the observer / modeller. However, the physical process described by a PDE is usually <u>independent</u> of the observer.

  $\Rightarrow$ PDEs which describe a real phenomenon should <u>first</u> be written in a way that is <u>independent of any coordinate system</u>.

<u>How?</u> $\Rightarrow$ use universal differential operators:

e.g.: $\nabla f$ or $\nabla \cdot \underline{A}$ or $\nabla \times \underline{A}$

<u>Example</u>: The Laplace Equation: $\boxed{\nabla^2 f = 0}$

$\Rightarrow$ this is a <u>universal</u> equation, so that the solution to this equation, whether expressed in Cartesian coordinates $(x, y, z)$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

or in spherical polar coordinates $(r, \theta, \phi)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} = 0$$

represents the <u>same scalar field $f$</u>

- For a list of standard differential operators expressed in various coordinate systems see Handout.

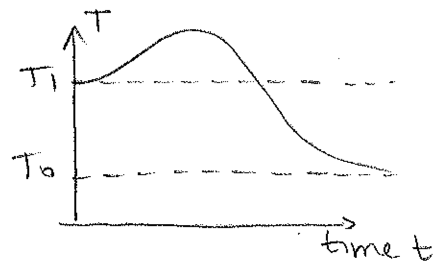
## b. Selection of a coordinate system

<u>For a given PDE</u>, a judicious selection of a coordinate system is usually very useful. The idea is to use a system which best represents the symmetries of the physical problem

→ to model a ball : use spherical coordinates
→ to model a cylinder/cone : use cylindrical coordinates
→ to model a cube/rectangle : use Cartesian coordinates

- Finally, it is always a good idea to use one's physical intuition of the system as a guide to find the solution or to critically assess the solution obtained.

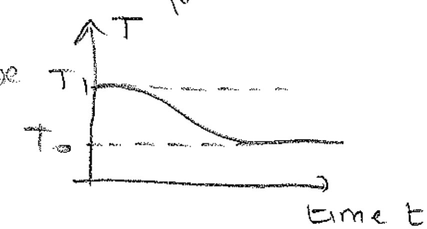
e.g. to model the cooling of a sphere initially at temperature $T_1$, immersed in a "bath" at temperature $T_0$ ⇒ we know that the sphere is unlikely to get hotter than $T_1$ at any time. ⇒ so the solution cannot be




⇐ NOT PHYSICAL

In fact, we also know that as $t \to \infty$, the temperature of the sphere should approach $T_0$

→ the solution is more likely to be




- In fact, we will see that for LINEAR PDEs there exists only 3 possible types of behaviour and these 3 are directly related to well-known physical systems
  ⇒ understanding the link between the mathematical nature of a PDE and its physical behaviour will help find solutions more easily

These are:
- the wave equation
- the heat equation (the diffusion equation)
- Laplace's equation

## (3) Fundamental 2nd order PDEs, (linear)

### a. The wave equation

- Generally written as

$$\frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f$$

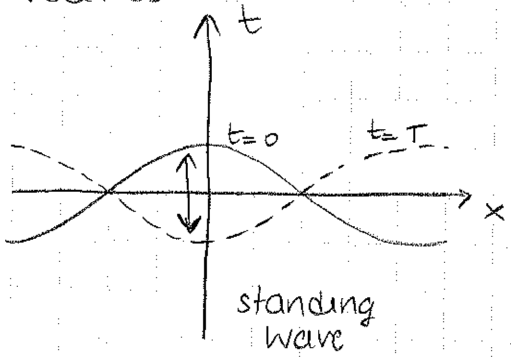
$\Rightarrow$ In cartesian coordinates for example

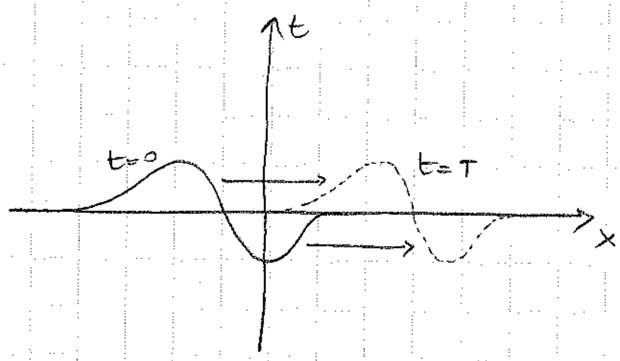
$$\frac{\partial^2 f}{\partial t^2} = c^2 \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right)$$

- The typical behaviour of the solution is oscillatory in time & space

- There is no dissipation (energy is conserved)

- There can be standing waves or propagating waves




Examples in Nature

- sound waves
- light (electromagnetic waves)
- seismic waves
- water waves

⋮

## 10. The heat equation (diffusion equation)

- Generally written as

$$\frac{\partial f}{\partial t} = \nabla \cdot ( k \nabla T ) = k \nabla^2 T \text{ if } k \text{ is constant}$$

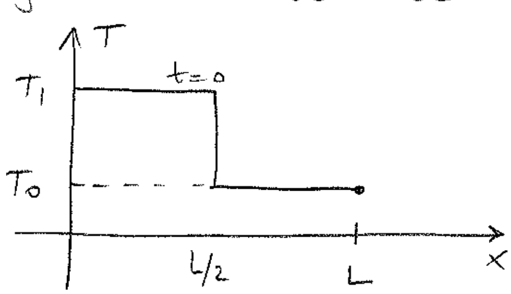
eg: in cartesian coordinates

$$\frac{\partial f}{\partial t} = k \left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right)$$

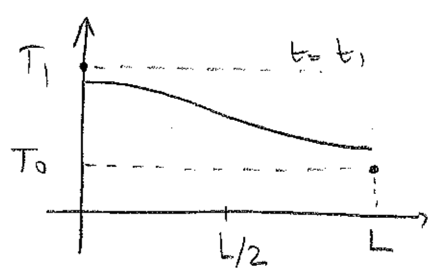
- This equation describes a diffusion process, its typical behaviour is to smooth out (and/or) dissipate gradients. The most commonly use example is the behaviour of a temperature field (hence the name "heat" equation).

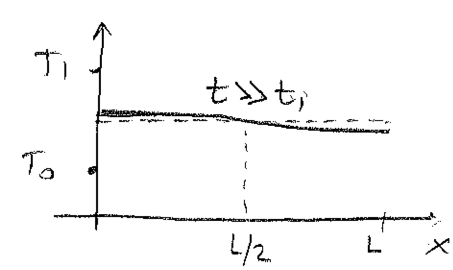
Example

If the initial condition for temperature in an insulated cylindrical thin rod of length L is




then



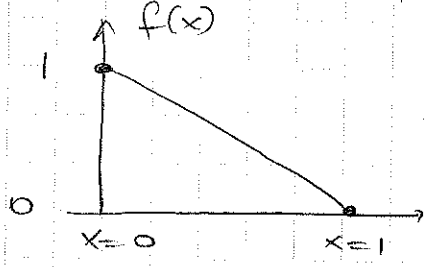

Physical intuition is very useful for diffusion problems.

## c. Laplace's equation

- Generally written as $\nabla^2 f = 0$
- Note that $\nabla^2 f = 0$ is the steady-state version of the heat equation $\Rightarrow$ can be thought of as the "end-product" of a diffusion process (after waiting an $\infty$ time)

Example: What is the solution to $\nabla^2 f = 0$ in 1D if $\begin{cases} f(x=0) = 1 \\ f(x=1) = 0 \end{cases}$

Idea: Imagine a 1-D rod of lenght 1, with one end held at temperature 1 and the other at temperature 0.
After an infinite time, the temperature in the rod has equilibrated to




$\rightarrow$ the solution to $\nabla^2 f = 0$ with these bcs.

Check: in 1D, $\nabla^2 f = f_{xx}$ so we simply solve

$f_{xx} = 0$ with above bcs

$\rightarrow$ solution is $f(x) = 1 - x$

## (4) Fundamental 1st order PDE: the transport equation

$$\boxed{\frac{\partial f}{\partial t} + \nabla \cdot (\underline{u} f) = 0}$$

where $\underline{u}$ is a velocity field (known)

Expression in Cartesian coordinates:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(u f) + \frac{\partial}{\partial y}(v f) + \frac{\partial}{\partial z}(w f) = 0$$

if

$$\underline{u} = (u, v, w)$$

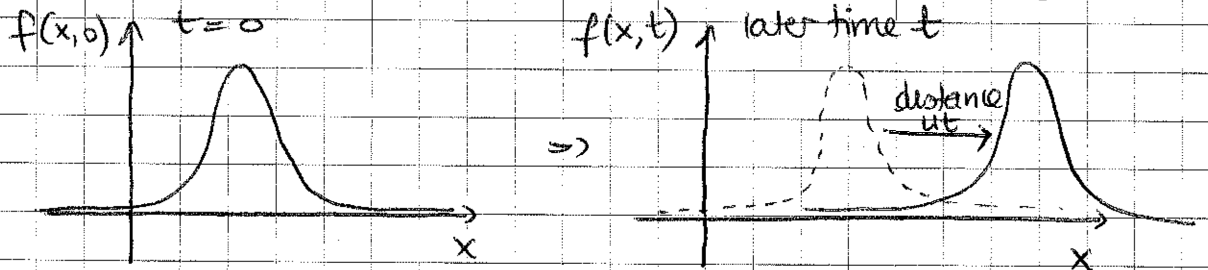
- This equation expresses the conservation of the quantity $f$ through transport by the velocity field $\underline{u}$.

- If $\underline{u}$ is a constant field then

$$\boxed{\frac{\partial f}{\partial t} + \underline{u} \cdot \nabla f = 0}$$

(note that this is also generally true for all vector fields $\underline{u}$ s.t. $\boxed{\nabla \cdot u = 0}$)

A solution is merely "moved around" by $\underline{u}$.




(note the similarity with propagation of wave).

See ppt for movies in more than 1D.

## (5) Additional conditions & well-posedness (a first look at)

- When modelling physical problems, the PDE is <u>always</u> accompanied by additional conditions, usually in the form of
  - initial conditions (for a time-dependent problem)
  - boundary conditions (for a problem on a finite domain)
  - regularity conditions (either, regularity/bound at infinity, or regularity at a coordinate singularity)

  The behaviour of a solution depends as much of the PDE than on these additional conditions

- For a given PDE with given additional conditions, there can be no, one or many possible solutions

  <u>example</u>: $u_t = u_x$

  - Given this PDE <u>without</u> any additional conditions, there are an infinity of solutions ($u = c$ for all $c \in \mathbb{R}$)
  - Given this PDE together with $u(x, t=0) = \phi(x)$ there is a unique solution (see Chapter 2).
  - Given this PDE with $u(x, t=x) = \phi(x)$ then there is no solution (see Chapter 2) or an infinite # of solutions.

- For a given equation and set of additional conditions, a small change in the parameters of the equation or of the conditions can lead to a <u>big</u> change in the solution.

Example $\quad u_t = -u_{xx} \quad t>0$
$\quad u(x,0) = 1$

→ obvious solution is $u(x,t) = 1$
but if we had chosen $u(x,0) = 1 + \frac{1}{n} \cdot \sin(nx)$
then the solution is

$$u(x,t) = 1 + \frac{1}{n} e^{n^2 t} \sin(nx) \qquad \text{(CHECK THIS)}$$

Now for $n$ large enough $1 + \frac{1}{n}\sin(nx) \simeq 1$
but after a time $t$ large enough $\frac{e^{n^2 t}}{n} \gg 1$
so a small difference in initial conditions creates an enormous difference in the final solution.

Definition:
a PDE (or set of PDE) and its associated additional conditions is a well-posed problem if
- it has a solution
- this solution is unique
- the structure of the solution is unchanged by infinitesimal variations of the parameters and/or of the additional conditions.

Typically: • a well-thought, well-modelled physical problem will result in a well-posed problem.
because in nature, the solution exists.
(although simplifying assumptions & shortcuts often lead to ill-posed problems)

BUT: not necessarily always the case. For nonlinear PDEs:
• multiple solutions can exist in which slight changes in the boundary conditions lead to one, or the other equilibrium
• sometimes, it is the discontinuous solutions that interest us (eg shock physics)
⇒ called weak solutions