PARTIAL DIFFERENTIAL EQUATIONS

CHAPTER I : Introduction & Reviews

I Partial differentiation (RHB Ch 5)

① Definitions and examples

- Given a function $f : \mathbb{R}^n \to \mathbb{R}$

$$(x_1, \dots x_n) \to f(x_1, \dots, x_n)$$

then the partial derivative of f with respect to x_i is

$$\frac{\partial f}{\partial x_i} = \lim_{\epsilon \to 0} \frac{f(x_1, \dots, x_i + \epsilon, \dots x_n) - f(x_1, \dots, x_i, \dots x_n)}{\epsilon}$$

- Notation & properties
  - typically we note $\frac{\partial f}{\partial x_i} = f_{x_i}$
  - we can similarly define higher-order derivatives

    e.g. $f_{x_i x_j} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$

  - The order of the partial derivatives is irrelevant:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

- Example : Let $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}} \qquad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}} \qquad \frac{\partial f}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{-xy}{(x^2+y^2+z^2)^{3/2}} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

(2) Chain rule and changes of variables

- If f is a function of the n variables $(x_1, \dots, x_n)$ and each x_i is a function of the m variables $(u_1, \dots u_m)$ then

$$\frac{\partial f}{\partial u_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial u_j} \quad \text{for } j = 1 \dots m$$

- This property can be used to change variables from one coordinate system to the next

- Example : Cartesian $\Leftrightarrow$ Polar

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{aligned} r &= \sqrt{x^2+y^2} \\ \theta &= \tan^{-1}(y/x) \end{aligned}$$

$$\left.\frac{\partial f}{\partial r}\right|_\theta = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial r}\right|_\theta + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial r}\right|_\theta$$

notation means at constant θ

$$= \cos\theta \left.\frac{\partial f}{\partial x}\right|_y + \sin\theta \left.\frac{\partial f}{\partial y}\right|_x$$

$$\left.\frac{\partial f}{\partial \theta}\right|_r = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial \theta}\right|_r + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial \theta}\right|_r$$

$$= -r\sin\theta \left.\frac{\partial f}{\partial x}\right|_y + r\cos\theta \left.\frac{\partial f}{\partial y}\right|_x$$

or, going the other way

$$\left.\frac{\partial f}{\partial x}\right|_y = \left.\frac{\partial f}{\partial r}\right|_\theta \left.\frac{\partial r}{\partial x}\right|_y + \left.\frac{\partial f}{\partial \theta}\right|_r \left.\frac{\partial \theta}{\partial x}\right|_y = \frac{x}{r}\left.\frac{\partial f}{\partial r}\right|_\theta - \frac{y}{x^2}\frac{1}{\frac{y^2}{x^2}+1}\left.\frac{\partial f}{\partial \theta}\right|_r$$

$$= \cos\theta \left.\frac{\partial f}{\partial r}\right|_\theta - \frac{\sin\theta}{r}\left.\frac{\partial f}{\partial \theta}\right|_r$$

III Review: First order (first degree) ODEs (RHB Ch 14)

Before attacking the problem of solving PDEs, let's review ODEs.

3.1 Standard forms

Let y be a function of the independent variable x only: $y(x)$

$y =$ dependent variable

First order ODEs for $y(x)$ can be written in the equivalent forms

$$\frac{dy}{dx} = F(x, y)$$

or

$$A(x,y)\,dx + B(x,y)\,dy = 0$$

$$\left(\text{with } F(x,y) = -\frac{A}{B}\right)$$

3.2 Types of equation & method of solution

3.2.1 Separable variables

Suppose $F(x,y) = f(x)\,g(y)$

or

$$A(x,y) = a(x)\,\alpha(y)$$
$$B(x,y) = b(x)\,\beta(y)$$

Then

$$\frac{dy}{dx} = F(x,y) \Rightarrow \frac{dy}{g(y)} = dx \cdot f(x)$$

which can be integrated respectively in y and x:

$$\int \frac{dy}{g(y)} = \int dx\, f(x).$$

→ Need to know your integrals; see handout

Example $$\frac{dy}{dx} = \frac{4y}{x(y-3)}$$

$$\Rightarrow \quad \frac{y-3}{4y}\, dy = \frac{dx}{x}$$

$$\Rightarrow \quad \int \frac{y-3}{4y}\, dy = \int \frac{dx}{x}$$

$$\Rightarrow \quad \frac{1}{4}y - \frac{3}{4}\ln y + K = \ln x$$

K: arbitrary constant of integration to be fitted to boundary condition

Note: Although a solution was found, it is not always possible to write it out easily as $y(x)$.

3.2.2 Exact equations

Suppose we try to solve the form

$$A(x,y)\,dx + B(x,y)\,dy = 0$$

In some cases, this may be the exact differential of a function $U(x,y)$:

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy$$

if it so happens that $\begin{cases} A(x,y) = \dfrac{\partial U}{\partial x} \\ B(x,y) = \dfrac{\partial U}{\partial y} \end{cases}$ then $dU = 0$

and the solution is simply $U = \text{constant}$.

How do we know this is an exact differential?

If $A = \dfrac{\partial U}{\partial x}$ and $B = \dfrac{\partial U}{\partial y}$ then $\boxed{\dfrac{\partial A}{\partial y} = \dfrac{\partial B}{\partial x}}$

$\Rightarrow$ Method:

Given the eqvation $A(x,y)dx + B(x,y)dy = 0$

① Test if exact: calculate $\frac{\partial A}{\partial y}$ and $\frac{\partial B}{\partial x}$

② If exact then find the function $U(x,y)$ such that

$$A(x,y) = \frac{\partial U}{\partial x}$$
$$B(x,y) = \frac{\partial U}{\partial y}$$

③ Set this function to an arbitrary constant to obtain solution

Example: $y\,dx + x\,dy = 0$ $\quad A(x,y) = y$, $B(x,y) = x$

① $\frac{\partial A}{\partial y} = 1 = \frac{\partial B}{\partial x} \Rightarrow$ this is an exact differential.

② What is $V(x,y)$ satisfying

$$\begin{cases} \frac{\partial U}{\partial x} = y & : \text{try } U = xy + \text{a function of } y \\ \frac{\partial U}{\partial y} = x & : \text{try } U = xy + \text{a function of } x \end{cases}$$

$\Rightarrow$ $U = xy + \text{constant}$

③ Set $U = \text{const}$ to get solution so

$U(x,y) = K$

$\Rightarrow \quad xy = K'$ (another constant)

$\Rightarrow \quad y = \frac{K'}{x}$

3.2.3 Linear equations

- Linear first order ODEs can always be written as

$$\frac{dy}{dx} + P(x)y = Q(x)$$

- Suppose we could find a function $\mu(x)$ such that

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)\,y = \frac{d}{dx}\big(\mu(x)y\big) \qquad (*)$$

Then the linear ODE would become

$$\frac{d}{dx}\big(\mu(x)y\big) = \mu(x)Q(x)$$

which can be formally integrated in x to give the solution

$$\mu(x)y(x) - \mu(0)y(0) = \int_0^x \mu(x')Q(x')\,dx'$$

- To find $\mu(x)$, we need to find a function satisfying $(*)$:

$$\cancel{\mu(x)\frac{dy}{dx}} + \mu(x)P(x)\,y = \cancel{\mu(x)\frac{dy}{dx}} + \frac{d\mu}{dx}y$$

$$\Rightarrow \quad \mu(x)P(x) = \frac{d\mu}{dx}$$

$$\Rightarrow \quad \frac{d\mu}{\mu} = P(x)\,dx$$

$$\Rightarrow \quad \ln\mu = \int P(x)\,dx$$

$$\Rightarrow \quad \boxed{\mu(x) = e^{\int P(x)dx}}$$

So: <u>Method</u>: ① Calculate $\mu(x) = e^{\int P(x)dx}$

② write $\frac{d}{dx}\big(\mu(x)y\big) = Q(x)\mu(x)$ and integrate it.

Example $\quad \dfrac{dy}{dx} + \dfrac{2-3x^2}{x^3}\, y = 1 \quad \Rightarrow \quad P(x) = \dfrac{2}{x^3} - \dfrac{3}{x}$

$Q(x) = 1$

① $$\mu(x) = e^{\int \left(\frac{2}{x^3} - \frac{3}{x}\right) dx}$$

$$= e^{-\frac{1}{x^2} - 3\ln x}$$

$$= e^{-\frac{1}{x^2}} \cdot \frac{1}{x^3}$$

② so $$\frac{d}{dx}\left(\frac{1}{x^3} e^{-\frac{1}{x^2}} y\right) = \frac{1}{x^3} e^{-\frac{1}{x^2}} \cdot 1$$

integrate in x: (say from $x' = 1$ to x, to avoid singularity at $x = 0$)

$$\frac{1}{x^3} e^{-\frac{1}{x^2}} y(x) - e^{-1} y(1) = \int_1^x \frac{1}{x'^3} e^{-\frac{1}{x'^2}}\, dx'$$

$$= \frac{1}{2}\left[e^{-\frac{1}{x'^2}}\right]_1^x$$

$$= \frac{1}{2}\left(e^{-\frac{1}{x^2}} - e^{-1}\right)$$

so $$y(x) = e^{-1}\left(y(1) - \frac{1}{2}\right) x^3 e^{\frac{1}{x^2}} + \frac{x^3}{2}$$

3.2.4 Inexact equations

(A similar idea: see notes in RHB).

3.2.5 Homogeneous equations

Homogeneous equations are ODEs that can be written as

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

They are suitably transformed into a simpler form by a change of variables:

$$v = \frac{y}{x} \quad (\text{or } y = vx) \quad \text{where } v = v(x)$$

<u>Indeed</u>: $$\frac{dy}{dx} = \frac{d}{dx}(vx) = x\frac{dv}{dx} + v$$
$$= F(v)$$

So $$x\frac{dv}{dx} + v = F(v)$$
$$\Rightarrow \frac{dv}{F(v)-v} = \frac{dx}{x}$$ which can be integrated in v and in x.

<u>Example</u> $$(y-x)\frac{dy}{dx} + (2x+3y) = 0$$

① Is it homogeneous? Yes: divide by x

$$\left(\frac{y}{x}-1\right)\frac{dy}{dx} + \left(2+\frac{3y}{x}\right) = 0$$

$$\frac{dy}{dx} + \frac{2+3\frac{y}{x}}{\frac{y}{x}-1} = 0$$

② Let $v = \frac{y}{x}$ then

$$x\frac{dv}{dx} + v + \frac{2+3v}{v-1} = 0$$

$$x\frac{dv}{dx} + \frac{(v-1)v + 2 + 3v}{v-1} = 0$$

$$x\frac{dv}{dx} + \frac{v^2+2v+2}{v-1} = 0 \rightarrow \underline{\text{Homework}}.$$