6.10. For the initial value problem for gas dynamics (6.15), (6.18), show that if the initial velocity distribution $f(x)$ is nondecreasing, shocks never develop for $t \geq 0$. On the other hand, if $f'(x) < 0$ over some interval of the x-axis, a shock eventually develops at some positive t. Follow the analysis of Problem 6.2 to derive formulas for the time and location of the first appearance of a shock.

7. The Method of Probability Generating Functions. Applications to a Trunking Problem in a Telephone Network and to the Control of a Tropical Disease

We discuss in this section an important application of first order linear partial differential equations to some problems in probability, namely to problems that arise in the study of certain processes known as stochastic processes. The material can be understood by students without background in probability. In the text we discuss two applications of the method of probability generating functions and in the problems we describe applications to Poisson, Yule, Polya, birth and death, and other stochastic processes. The main references for the material of this section are Feller[7] Chapter XVII, Sections 5–7 and Chiang,[8] Chapters 2 and 3.

A Trunking Problem in a Telephone Network

We consider an idealized telephone network consisting of an infinite number of lines (trunklines), and assume that calls originate and terminate within the network during the time interval $[0, \infty)$ according to certain hypotheses which we describe below. The problem to be solved is the following. Given any non-negative integer n, find the probability $P_n(t)$ that exactly n lines are in use at time t, $0 < t < \infty$, assuming that the initial probabilities $P_n(0)$, $0 \leq n < \infty$, are known.

In stating the hypotheses concerning the initiation (birth) and termination (death) of phone calls within the network we will use the symbol $o(h)$ to denote any quantity which vanishes more rapidly than h as $h \to 0$; i.e., $\lim_{h\to 0} [o(h)/h] = 0$. The reasonableness and validity of the hypotheses are discussed in the book of Feller.[7] The hypotheses are: (i) if a line is occupied at time t, the probability of the conversation ending during the time interval $(t, t + h)$ is $\mu h + o(h)$, where μ is a constant; (ii) the probability of a call starting during the interval $(t, t + h)$ is $\lambda h + o(h)$, where λ is a constant; and (iii) the probability of two or more changes occurring (calls starting or ending) during the interval $(t, t + h)$ is $o(h)$.

The first step in the determination of the probabilities $P_n(t)$ consists of deriving a system of ordinary differential equations that are satisfied by $P_n(t)$. Let us suppose for a moment that t is fixed and that the probabilities $P_n(t)$ are known for all n, $0 \leq n < \infty$, and let us try to determine $P_n(t + h)$, the probability that n lines are in use at time $t + h$. Suppose first that $n \geq 1$. A moment's reflection should convince the reader that there will be n lines in use at time $t + h$ only if one of the following conditions is satisfied: (1) at time t, $n - 1$ lines are in use and one call originates during the time interval $(t, t + h)$; (2) at time t, $n + 1$ lines are in use and one call terminates during the interval $(t, t + h)$; (3) at time t, n lines are in use and no change occurs in the network during the interval $(t, t + h)$; and (4) two or more changes occur during the interval $(t, t + h)$. According to our hypotheses, the probability of the last event (4) is $o(h)$ while the probability of (1) is

$$[\lambda h + o(h)]P_{n-1}(t),$$

the probability of (2) is

$$(n + 1)[\mu h + o(h)]P_{n+1}(t),$$

and the probability of (3) is

$$[1 - \lambda h - n\mu h - o(h)]P_n(t).$$

Since the contingencies (1), (2) and (3) are mutually exclusive, their probabilities add. Therefore

$$P_n(t + h) = \lambda h P_{n-1}(t) + (n + 1)\mu h P_{n+1}(t) + (1 - \lambda h - n\mu h)P_n(t) + o(h). \tag{7.1}$$

Using (7.1) to form the difference quotient $[P_n(t + h) - P_n(t)]/h$ and letting $h \to 0$, we obtain the ordinary differential equations

$$P_n'(t) = -(\lambda + n\mu)P_n(t) + \lambda P_{n-1}(t) + (n + 1)\mu P_{n+1}(t), \tag{7.2}$$

which must hold for all $n \geq 1$ and $0 < t < \infty$. For $n = 0$ a similar analysis leads to the equation

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t). \tag{7.3}$$

Since the initial probabilities $P_n(0)$, $0 \leq n < \infty$, are assumed to be known, the problem of finding the probabilities $P_n(t)$ for all $t > 0$ has been reduced to the initial value problem for the infinite system of ordinary differential equations (7.2), (7.3). The question of existence and uniqueness of solution of this initial value problem is not easy; the interested student can read the discussion and references cited in Feller.[7] Here, we describe a method for finding the solution of the problem by solving an initial value problem for a first order linear partial differential equation.

The function

$$G(t,s) = \sum_{n=0}^{\infty} P_n(t)s^n \tag{7.4}$$

is known as the *probability generating function* for the probabilities $P_n(t)$. As a consequence of the system of o.d.e.'s (7.2) it is easy to show that $G(t, s)$ must satisfy a linear first order partial differential equation. In fact, differentiating (7.4) we get

$$\frac{\partial G}{\partial s} = \sum_{n=1}^{\infty} nP_n(t)s^{n-1} = \sum_{n=0}^{\infty} (n + 1)P_{n+1}(t)s^n, \tag{7.5}$$

$$\frac{\partial G}{\partial t} = \sum_{n=0}^{\infty} P_n'(t)s^n. \tag{7.6}$$

Substitution of the expressions (7.2), (7.3) for $P_n'(t)$ into (7.6), followed by rearrangement and identification of the resulting series with the series in (7.4) and (7.5) yields the p.d.e. for G,

$$\frac{\partial G}{\partial t} + \mu(s-1)\frac{\partial G}{\partial s} = \lambda(s-1)G. \tag{7.7}$$

On the other hand, knowledge of the initial probabilities $P_n(0)$ leads to the initial condition for G along the line $t = 0$ of the (t, s)-plane,

$$G(0,s) = g(s), \tag{7.8}$$

where

$$g(s) = \sum_{n=0}^{\infty} P_n(0)s^n. \tag{7.9}$$

It is easy to obtain the solution of the initial value problem (7.7), (7.8). The associated system of o.d.e.'s of (7.7) is

$$\frac{dt}{1} = \frac{ds}{\mu(s-1)} = \frac{dG}{\lambda(s-1)G},$$

and two functionally independent first integrals are

$$u_1 = e^{-\mu t}(s-1), \qquad u_2 = e^{-\frac{\lambda}{\mu}s}G. \tag{7.10}$$

Since u_1 does not depend on G, the general integral of (7.7) is

$$u_2 = f(u_1)$$

where f is an arbitrary C^1 function of a single variable. Substituting (7.10) in the general integral and solving for G we obtain the solutions of (7.7),

$$G(t, s) = e^{\frac{\lambda}{\mu}s} f(e^{-\mu t}(s-1)). \tag{7.11}$$

The initial condition (7.8) determines the function f. In fact setting $t = 0$ in (7.11) and using (7.8) yields

$$g(s) = e^{\frac{\lambda}{\mu}s} f(s-1),$$

and, consequently,

$$f(s) = g(s+1)e^{-\frac{\lambda}{\mu}(s+1)}. \tag{7.12}$$

Finally, substituting (7.12) into (7.11) and simplifying we obtain the solution of the initial value problem (7.7), (7.8)

$$G(t, s) = g(1 + e^{-\mu t}(s-1)) \exp\left[\frac{\lambda}{\mu}(s-1)(1-e^{-\mu t})\right]. \tag{7.13}$$

Once the probability generating function $G(t, s)$ has been found, the probabilities $P_n(t)$ can be found from the familiar formula for the coefficients of the Taylor series (7.4),

$$P_n(t) = \frac{1}{n!}\left[\frac{\partial^n}{\partial s^n} G(t, s)\right]_{s=0} \tag{7.14}$$

or by obtaining the series expansion of $G(t, s)$ in powers of s by some other means and then identifying the coefficients of the series with $P_n(t)$.

In order to illustrate the method of probability generating functions (p.g.f.), let us suppose that exactly one line is in use at time $t = 0$. This means that

$$P_1(0) = 1 \quad \text{and} \quad P_n(0) = 0 \quad \text{for} \quad n \neq 1, \tag{7.15}$$

and, therefore,

$$g(s) = \sum_{n=0}^{\infty} P_n(0)s^n = s. \tag{7.16}$$

Substitution of (7.16) into (7.13) yields the p.g.f.

$$G(t, s) = [1 + e^{-\mu t}(s-1)] \exp\left[\frac{\lambda}{\mu}(s-1)(1-e^{-\mu t})\right]. \tag{7.17}$$

The probabilities $P_n(t)$ can be determined using formula (7.14). For $n = 0$ and $n = 1$ we have

$$P_0(t) = G(t, 0) = (1 - e^{-\mu t}) \exp\left[\frac{\lambda}{\mu}(e^{-\mu t} - 1)\right],$$

$$P_1(t) = \frac{\partial G}{\partial s}(t, 0) = \left[e^{-\mu t} + \frac{\lambda}{\mu}(1 - e^{-\mu t})^2\right] \exp\left[\frac{\lambda}{\mu}(e^{-\mu t} - 1)\right].$$

Clearly the computational labor to obtain $P_n(t)$ increases rapidly with n. An alternate method for obtaining $P_n(t)$ directly from (7.17) is outlined in Problem 7.4.

A more realistic model of a telephone network with a finite number of lines can be analyzed in a similar way. For details see Feller[7] Chapter XVII, Section 7.

A Problem in the Control of a Tropical Disease

Schistosomiasis is a parasitic infection that is estimated to affect more than two hundred million people in tropical and subtropical countries of the world. It is characterized by long term debility which is thought by many to be a significant obstacle to the advancement of many underdeveloped countries where large segments of the population are more or less permanently infected. The persistence of this infection in a locality depends on a complex cycle of events involving humans, certain parasitic flatworms (schistosomes) and particular species of snails. A probabilistic study of this cycle of events has been carried out in a paper by Nasell and Hirsch.[9] The results of this study make possible the comparison of the relative effectiveness of various procedures aimed at control or eradication of the disease. We present here a problem that appears in the paper[9] concerning the determination of a certain probability generating function.

The probability generating function $G(t, s)$ must satisfy the p.d.e.

$$\frac{\partial G}{\partial t} + \mu(s-1)\frac{\partial G}{\partial s} = \frac{1}{2}\nu Y(t)(s-1)G \tag{7.18}$$

and the initial condition

$$G(t_0, s) = s^m \tag{7.19}$$

along the line $t = t_0$ of the (t,s) plane. $Y(t)$ is a given continuous function, μ and ν are constants and m is a nonnegative integer. It is an easy exercise (see Problem 7.8) to obtain the first integrals of (7.18),

$$u_1 = e^{-\mu t}(s-1), \qquad u_2 = Ge^{-\frac{1}{2}\beta(t)(s-1)}, \tag{7.20}$$

where

$$\beta(t) = e^{-\mu t}\int_0^t Y(\tau)e^{\mu\tau}\,d\tau.$$

Now, the general integral of (7.18) is

$$Ge^{-\frac{1}{2}\beta(t)(s-1)} = f(e^{-\mu t}(s-1)), \tag{7.21}$$

where f is an arbitrary C^1 function. Solving (7.21) for G we obtain the solutions of (7.18),

$$G(t, s) = e^{\frac{1}{2}\beta(t)(s-1)}f(e^{-\mu t}(s-1)). \tag{7.22}$$

The initial condition (7.19) determines the function f since it requires that

$$s^m = e^{\frac{1}{2}\beta(t_0)(s-1)}f(e^{-\mu t_0}(s-1)). \tag{7.23}$$

Setting $z = e^{-\mu t_0}(s-1)$ we have $s = 1 + ze^{\mu t_0}$ and (7.23) yields

$$f(z) = (1 + ze^{\mu t_0})^m \exp\left[-\frac{1}{2}\beta(t_0)ze^{\mu t_0}\right].$$

Therefore

$$f(e^{-\mu t}(s-1)) = [1 + e^{-\mu(t-t_0)}(s-1)]^m \cdot\exp\left[-\frac{1}{2}\beta(t_0)e^{-\mu(t-t_0)}(s-1)\right],$$

and substituting in (7.22) we obtain the solution of the initial value problem (7.18), (7.19),

$$G(t,s) = [1 + e^{-\mu(t-t_0)}(s-1)]^m \cdot\exp\left\{\frac{1}{2}[\beta(t) - \beta(t_0)e^{-\mu(t-t_0)}](s-1)\right\}. \tag{7.24}$$

Problems

7.1. Derive equation (7.3).

7.2. Derive (7.7) from (7.4), (7.5) and (7.6). [Note that (7.5) expresses $\partial G/\partial s$ in two ways.]

7.3. Derive (7.13) from (7.11) and (7.12).

7.4. For fixed t, equation (7.17) has the form

$$G(t,s) = (a + bs)e^{c+ds}$$

where a, b, c, d are constants depending on t. Expand G in a Taylor series in s using the expansion for e^s and derive the formulas

$$P_n(t) = \frac{1}{n!}\frac{1}{\mu}\left(\frac{\lambda}{\mu}\right)^{n-1}(1 - e^{-\mu t})^{n-1}[\lambda(1 - e^{-\mu t})^2 + n\mu e^{-\mu t}] \times \exp\left[\frac{\lambda}{\mu}(e^{-\mu t} - 1)\right],$$

for $0 \le t < \infty$ and $n = 0, 1, 2, \ldots$.

7.5. For the telephone network discussed in this section find the probability generating function and the probabilities $P_n(t)$ if at time $t = 0$, (a) two lines are in use, (b) m lines are in use, where m is a positive integer.

7.6. The *expectation* (mean value) $E(t)$ of the number of telephone lines in use at time t is defined by

$$E(t) = \sum_{n=0}^{\infty} nP_n(t).$$

It is a weighted mean of the number of lines that may be in use at time t, weighted by the corresponding probabilities. Show that

$$E(t) = \left.\frac{\partial G(t,s)}{\partial s}\right|_{s=1},$$

and calculate $E(t)$ if one line is in use at time $t = 0$.

7.7. In the formula of Problem 7.4 let $t \to \infty$ to show that

$$\lim_{t\to\infty} P_n(t) = \frac{e^{-\frac{\lambda}{\mu}}\left(\frac{\lambda}{\mu}\right)^n}{n!},$$

which is the Poisson distribution with parameter λ/μ.

7.8. Derive the first integrals (7.20) of equation (7.18).

7.9. *The Poisson process*. In many physical processes the occurrence of an event at a particular moment is independent of time and of the

number of events that have already taken place. Examples are accidents occurring in a city, the splitting of atoms of a radioactive substance, breakage of chromosomes under harmful irradiation and phone calls arriving at a switchboard. Let $X(t)$ denote the total number of events occurring during the time interval $(0, t)$ and let $P_n(t)$ denote the probability that $X(t) = n$. A process is said to be a Poisson process if, for any $t \geq 0$, (i) the probability that an event occurs during the interval $(t, t+h)$ is $\lambda h + o(h)$ where λ is a constant, and (ii) the probability that more than one event occurs during $(t, t+h)$ is $o(h)$.
(a) Derive the system of o.d.e.'s

$$P_0'(t) = -\lambda P_0(t)$$

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t), \qquad n = 1, 2, \ldots.$$

(b) Show that the p.g.f. $G(t,s) = \sum P_n(t)s^n$ satisfies the initial value problem,

$$\frac{\partial G}{\partial t} = -\lambda(1 - s)G$$

$$G(0, s) = 1.$$

(Note that $P_0(0) = 1$ and $P_n(0) = 0$ for $n = 1, 2, \ldots$.)
(c) Solve the initial value problem for G to show

$$G(t,s) = e^{-\lambda t(1-s)},$$

and obtain the probabilities

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \qquad n = 0, 1, 2, \ldots$$

(This is the Poisson distribution with parameter λt.)
(d) Show that the expectation $E(t) = \lambda t$ (see Problem 7.6).

7.10. *The time-dependent Poisson process.* This is a Poisson process in which λ is not constant but is instead a function of t, $\lambda = \lambda(t)$. Follow the instructions of Problem 7.9 for this case and show that the formulas for $G(t, s)$, $P_n(t)$, and $E(t)$ are obtained from those of Problem 7.9 by replacing λt with $\int_0^t \lambda(\tau)d\tau$.

7.11. *The Yule process.* This process was first studied by Yule in connection with the mathematical theory of evolution. It is a simple example of what is known as a *pure birth process*. Consider a population of members (such as bacteria) which give birth to new members but do not die (bacteria may do this by splitting). Assume that during any short time interval $(t, t + h)$ each member has probability $\lambda h + o(h)$ to create a new member (λ is a constant), and that members give birth independently of each other. Then, if at time t the population size is n, the probability of increase of the population by exactly one during $(t, t + h)$ is $n\lambda h + o(h)$. Assume also that at time $t = 0$ the population size is n_0, so that if $P_n(t)$ is the probability that the population size is n at time t, then

$$P_{n_0}(0) = 1 \quad \text{and} \quad P_n(0) = 0 \quad \text{for} \quad n \neq n_0.$$

(a) Derive the system of o.d.e.'s,

$$P_{n_0}'(t) = -n_0\lambda P_{n_0}(t)$$

$$P_n'(t) = -n\lambda P_n(t) + (n - 1)\lambda P_{n-1}(t), \qquad n > n_0.$$

(b) Introduce the p.g.f. $G(t,s) = \sum_{n=n_0}^{\infty} P_n(t)s^n$ and show that it must satisfy the initial value problem

$$\frac{\partial G}{\partial t} + \lambda s(1 - s)\frac{\partial G}{\partial s} = 0$$

$$G(0, s) = s^{n_0}$$

(c) Show that

$$G(t, s) = s^{n_0}\left[\frac{e^{-\lambda t}}{1 - s(1 - e^{-\lambda t})}\right]^{n_0}$$

(d) Show that the expectation $E(t) = \sum_{n=n_0}^{\infty} nP_n(t)$ is

$$E(t) = n_0e^{\lambda t}.$$

This is the familiar exponential population growth.

7.12. *The time-dependent Yule process.* This is a Yule process in which λ is a function of t, $\lambda = \lambda(t)$. Follow the instructions of Problem 7.11 for this case and show that the formulas for $G(t, s)$ and $E(t)$ are obtained from those of Problem 7.11 by replacing λt with $\int_0^t \lambda(\tau)d\tau$.

7.13. *The Polya process.* This is a pure birth process for which it is assumed that if at time t the population size is n, the probability of increase of the population by exactly one during the interval $(t, t + h)$ is $\lambda_n(t)h + o(h)$ where

$$\lambda_n(t) = \frac{\lambda + \lambda an}{1 + \lambda at},$$

with λ and a constants. Note that $a = 0$ corresponds to a Poisson process. Assume that initially the population size is n_0.
(a) Derive the system of o.d.e.'s

$$P_{n_0}'(t) = -\frac{\lambda + \lambda an_0}{1 + \lambda at}P_{n_0}(t)$$

$$P_n'(t) = -\frac{\lambda + \lambda an}{1 + \lambda at}P_n(t) + \frac{\lambda + \lambda a(n - 1)}{1 + \lambda at}P_{n-1}(t), \; n > n_0.$$

(b) Introduce the p.g.f. $G(t, s) = \sum_{n=n_0}^{\infty} \text{p}\, P_n(t)s^n$ and show that it must satisfy the initial value problem

$$(1 + \lambda at)\frac{\partial G}{\partial t} + \lambda as(1 - s)\frac{\partial G}{\partial s} = -\lambda(1 - s)G$$

$$G(0, s) = s^{n_0}.$$

(c) Show that

$$G(t,s) = \left(\frac{s}{1 + \lambda at - \lambda ats}\right)^{n_0}(1 + \lambda at - \lambda ats)^{-1/a}$$

7.14. *The birth and death process.* This process allows for a population to decline as well as to grow, and therefore it provides a more realistic model for biological problems. Let $X(t)$ denote the size of population at time t and $P_n(t)$ denote the probability that $X(t) = n$. The basic hypotheses are the following: If $X(t) = n$, then during the interval $(t, t + h)$: (i) the probability of one birth occurring is $\lambda_n(t)h + o(h)$; (ii) the probability of one death occurring is $\mu_n(t)h + o(h)$; and (iii) the probability of more than one change (birth or death) is $o(h)$. Show that the $P_n(t)$ must satisfy the system of o.d.e.'s,

$$P_0'(t) = -[\lambda_0(t) + \mu_0(t)]P_0(t) + \mu_1(t)P_1(t)$$

$$P_n'(t) = -[\lambda_n(t) + \mu_n(t)]P_n(t)$$

$$+ \lambda_{n-1}(t)P_{n-1}(t) + \mu_{n+1}(t)P_{n+1}(t),\ n \geq 1.$$

7.15. In the telephone network discussed in this section the process of initiation and termination of calls may be considered as a birth and death process with the population size being the number of lines in use. Show that in this case $\lambda_n(t) = \lambda$ and $\mu_n(t) = n$ and verify that the system of o.d.e.'s of Problem 7.14 becomes system (7.2), (7.3).

7.16. *A birth and death process with linear growth.* Consider a population of living elements, such as bacteria, that can split or die. During any short time interval $(t, t + h)$ the probability of any living element splitting into two is $\lambda h + o(h)$ and the probability of it dying is $\mu h + o(h)$, where λ and μ are constants. Assume that at $t = 0$ the population size is n_0.

(a) Show that in the notation of Problem 7.14, $\lambda_n(t) = n\lambda$ and $\mu_n(t) = n\mu$ and write down the system of o.d.e.'s and the initial conditions for the $P_n(t)$.

(b) Show that the p.g.f. $G(t, s)$ must satisfy the initial value problem,

$$\frac{\partial G}{\partial t} + (1 - s)(\lambda s - \mu)\frac{\partial G}{\partial s} = 0$$

$$G(0, s) = s^{n_0}.$$

(c) If $\lambda \neq \mu$ obtain the solution

$$G(t, s) = \left[\frac{(\lambda s - \mu) + \mu(1 - s)e^{(\lambda-\mu)t}}{(\lambda s - \mu) + \lambda(1 - s)e^{(\lambda-\mu)t}}\right]^{n_0}$$

and show that the expectation is

$$E(t) = n_0 e^{(\lambda-\mu)t}.$$

(d) If $\lambda = \mu$ obtain the solution

$$G(t,s) = \left\{\frac{\alpha(t) + [1 - 2\alpha(t)]s}{1 - \alpha(t)s}\right\}^{n_0}$$

where $\alpha(t) = \lambda t/(1 + \lambda t)$, and show that the expectation is

$$E(t) = n_0.$$

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