Homework 3

Problem 1: Quasilinear equation. Find the solution of the PDE

$$xuu_x + yuu_y = u^2 - 1$$
$$u(x, x^2) = x^3 \text{ for } x > 0$$

and discuss (using the transversality condition) what happens at $x = 0$.

Problem 2: Traffic flow

- Complete the lecture notes by looking at the traffic flow problem with an initial velocity profile with $\frac{U_{\max}}{2} < u(x, 0) < U_{\max}$.

- Invent another possible flux law for the traffic flow (i.e. propose a new $V(N)$) and discuss the behavior of the solutions.

Problem 3:

Consider Euler's equation

$$\begin{aligned} u_t + uu_x &= 0 \\ u(x, 0) &= x \text{ for } x > 0 \\ u(x, 0) &= 0 \text{ for } x \leq 0 \end{aligned} \tag{1}$$

- What are the characteristic equations? Draw the characteristics.
- Find the solution $u(x, t)$?
- What is the domain of definition of the function? Where is the function continuous? Differentiable?

Now consider, for all times $t \geq 0$,

$$\begin{aligned} u_t + uu_x &= 0 \\ u(x, 0) &= 0 \text{ for } x \geq 1 \\ u(x, 0) &= x \text{ for } x \in (0, 1) \\ u(x, 0) &= 0 \text{ for } x \leq 0 \end{aligned} \tag{2}$$

- What are the characteristic equations? Draw the characteristics.
- Why does this solution involve a shock? Where/when does the shock begin?
- What is the equation governing the shock propagation? Solve this equation and determine the shock front $x = \gamma(t)$.

- Complete the problem by drawing the shock front on the (x, t) plane, and the characteristics. Write the solution $u(x, t)$ for all x, for $t > 0$.

Problem 4:

Consider the problem

$$u_t + tuu_x = 0$$
$$u(x, 0) = \sin(x) \text{ for all } x$$

- What are the characteristic equations? Draw the characteristics.
- By considering intersecting characteristics, show that the shock fronts occur at $t_c = \sqrt{2}$ and $x_c = \pi + 2k\pi$ for all integer values of k.
- Solve the problem for $t \in [0, t_c)$. (Note: you can leave the solution in an implicit form).
- Now realize that there exists an easy change of variable that will transform this equation into a conservative system for $t > 0$. What is it?
- By symmetry arguments, or otherwise, show that the shock fronts satisfy the equation $\mathrm{d}\gamma/\mathrm{d}t =$, and therefore complete the problem by giving a solution valid at all times. (Note: you can leave the solution in an implicit form).

Problem 5: Method of characteristics for multi-dimensional first-order PDEs.

Solve the two-dimensional transport equation:

$$u_t + c_1 u_x + c_2 u_y + \alpha u = 0$$
$$u(x, y, 0) = u_0(x, y) \text{ for all } x, y$$

where c_1 and c_2 are constants.

Problem 6: Application to information flow

Problems 7.1 - 7.7 in Phone Lines handout