EXERCISES 2.3

2.3.1. For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

*(a) $\dfrac{\partial u}{\partial t} = \dfrac{k}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial u}{\partial r}\right)$

(b) $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} - v_0\dfrac{\partial u}{\partial x}$

*(c) $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

(d) $\dfrac{\partial u}{\partial t} = \dfrac{k}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial u}{\partial r}\right)$

*(e) $\dfrac{\partial u}{\partial t} = k\dfrac{\partial^4 u}{\partial x^4}$

*(f) $\dfrac{\partial^2 u}{\partial t^2} = c^2\dfrac{\partial^2 u}{\partial x^2}$

2.3.2. Consider the differential equation

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0.$$

Determine the eigenvalues λ (and corresponding eigenfunctions) if ϕ satisfies the following boundary conditions. Analyze three cases ($\lambda > 0, \lambda = 0, \lambda < 0$). You may assume that the eigenvalues are real.

(a) $\phi(0) = 0$ and $\phi(\pi) = 0$

*(b) $\phi(0) = 0$ and $\phi(1) = 0$

(c) $\dfrac{d\phi}{dx}(0) = 0$ and $\dfrac{d\phi}{dx}(L) = 0$ (If necessary, see Sec. 2.4.1.)

*(d) $\phi(0) = 0$ and $\dfrac{d\phi}{dx}(L) = 0$

(e) $\dfrac{d\phi}{dx}(0) = 0$ and $\phi(L) = 0$

*(f) $\phi(a) = 0$ and $\phi(b) = 0$ (You may assume that $\lambda > 0$.)

(g) $\phi(0) = 0$ and $\dfrac{d\phi}{dx}(L) + \phi(L) = 0$ (If necessary, see Sec. 5.8.)

2.3.3. Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

Solve the initial value problem if the temperature is initially

(a) $u(x,0) = 6\sin\frac{9\pi x}{L}$

(b) $u(x,0) = 3\sin\frac{\pi x}{L} - \sin\frac{3\pi x}{L}$

*(c) $u(x,0) = 2\cos\frac{3\pi x}{L}$

(d) $u(x,0) = \begin{cases} 1 & 0 < x \le L/2 \\ 2 & L/2 < x < L \end{cases}$

[Your answer in part (c) may involve certain integrals that do not need to be evaluated.]

2.3.4. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2},$$

subject to $u(0, t) = 0, u(L, t) = 0$, and $u(x, 0) = f(x)$.

*(a) What is the total heat energy in the rod as a function of time?

(b) What is the flow of heat energy out of the rod at $x = 0$? at $x = L$?

*(c) What relationship should exist between parts (a) and (b)?

2.3.5. Evaluate (be careful if $n = m$)

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx \qquad \text{for } n > 0, m > 0.$$

Use the trigonometric identity

$$\sin a \sin b = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right].$$

*2.3.6. Evaluate

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx \qquad \text{for } n \geq 0, m \geq 0.$$

Use the trigonometric identity

$$\cos a \cos b = \frac{1}{2}\left[\cos(a+b) + \cos(a-b)\right].$$

(Be careful if $a - b = 0$ or $a + b = 0$.)

2.3.7. Consider the following boundary value problem (if necessary, see Sec. 2.4.1):

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ with } \frac{\partial u}{\partial x}(0, t) = 0, \frac{\partial u}{\partial x}(L, t) = 0, \text{ and } u(x, 0) = f(x).$$

(a) Give a one-sentence physical interpretation of this problem.

(b) Solve by the method of separation of variables. First show that there are no separated solutions which exponentially grow in time. [*Hint:* The answer is

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n e^{-\lambda_n k t}\cos\frac{n\pi x}{L}.\Big]$$

What is λ_n?

(c) Show that the initial condition, $u(x,0) = f(x)$, is satisfied if

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}.$$

(d) Using Exercise 2.3.6, solve for A_0 and $A_n (n \geq 1)$.

(e) What happens to the temperature distribution as $t \to \infty$? Show that it approaches the steady-state temperature distribution (see Sec. 1.4).

*2.3.8. Consider

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} - \alpha u.$$

This corresponds to a one-dimensional rod either with heat loss through the lateral sides with outside temperature 0° ($\alpha > 0$, see Exercise 1.2.4) or with insulated lateral sides with a heat sink proportional to the temperature. Suppose that the boundary conditions are

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0.$$

(a) What are the possible equilibrium temperature distributions if $\alpha > 0$?

(b) Solve the time-dependent problem $[u(x,0) = f(x)]$ if $\alpha > 0$. Analyze the temperature for large time ($t \to \infty$) and compare to part (a).

*2.3.9. Redo Exercise 2.3.8 if $\alpha < 0$. [Be especially careful if $-\alpha/k = (n\pi/L)^2$.]

2.3.10. For two- and three-dimensional vectors, the fundamental property of dot products, $\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}| \cos\theta$, implies that

$$|\boldsymbol{A} \cdot \boldsymbol{B}| \leq |\boldsymbol{A}||\boldsymbol{B}|. \tag{2.3.44}$$

In this exercise we generalize this to n-dimensional vectors and functions, in which case (2.3.44) is known as **Schwarz's inequality**. [The names of Cauchy and Buniakovsky are also associated with (2.3.44).]

(a) Show that $|\boldsymbol{A} - \gamma \boldsymbol{B}|^2 > 0$ implies (2.3.44), where $\gamma = \boldsymbol{A} \cdot \boldsymbol{B} / \boldsymbol{B} \cdot \boldsymbol{B}$.

(b) Express the inequality using both

$$\boldsymbol{A} \cdot \boldsymbol{B} = \sum_{n=1}^{\infty} a_n b_n = \sum_{n=1}^{\infty} a_n c_n \frac{b_n}{c_n}.$$

*(c) Generalize (2.3.44) to functions. [*Hint*: Let $\boldsymbol{A} \cdot \boldsymbol{B}$ mean the integral $\int_0^L A(x)B(x)\,dx$.]

2.3.11. Solve Laplace's equation inside a rectangle:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

subject to the boundary conditions

$$\begin{aligned} u(0,y) &= g(y) \qquad & u(x,0) &= 0 \\ u(L,y) &= 0 \qquad & u(x,H) &= 0. \end{aligned}$$

(*Hint*: If necessary, see Sec. 2.5.1.)