Take Home Midterm

If you think there is a typo, please email/call me asap (459-1055). To be handed in on Thursday, Feb. 10, noon. You need to justify all your answers. Answers without justifications will be counted as wrong. You **may not** ask for help from any other living being aside from the instructor. If you need help from the instructor, make sure to ask **before** Thurday, noon. Help will be limited to clarifications on the questions if needed.

Problem 1:

Problem 7.7.4 (b) of textbook. In addition to the questions asked, (1) calculate the 5 lowest eigenfrequencies and (2) hand in computer-generated plots of the modes corresponding to the 5 lowest eigenfrequencies.

Problem 2:

Problem 8.3.5 of textbook.

Problem 3:

A new planet has been discovered orbiting very close to its host star, so close in fact that it is tidally locked and always facing the star the same way[1]. As a result, there is a large temperature difference between the day-side and the night-side of the planet. The ultimate goal of this problem is to evaluate this day-to-night side surface temperature difference, when the planet is in thermal equilibrium.

In all that follows, we will be using a spherical coordinate system (r, θ, ϕ) where the $\theta = 0$ axis is the line connecting the center of the planet to the center of the star. As a result, $\theta = 0$ is the center of the day-side, and $\theta = \pi$ is the center of the night-side.

This planet is composed of a molten inner core, from $r = 0$ to $r = aR$ (where a is between 0 and 1), surrounded by a solid shell from $r = aR$ to $r = R$ (R is the radius of the planet).

Radioactive decay within the inner core provides a constant source of heat, which, in equilibrium, implies a constant *local* heat flux f_0 through the base of the solid shell. Mathematically, this implies

$$-k\frac{\partial T}{\partial r}\bigg|_{r=aR} = f_0 \tag{1}$$

where k is the (known) diffusivity of the shell, and f_0 is the (known) flux.

Moreover, the *total* heat flux received from the central star (as integrated over the surface of the planet) is $F_\star$. Note that $F_\star$ is a known quantity, which simply depends on the luminosity of the star and how far the planet is.

[1]The same thing happens with the Moon, which is always showing the same side to the Earth.

In equilibrium, the temperature profile within the shell satisfies Laplace's equation. Meanwhile, the surface of the planet has a temperature profile

$$T(R, \theta, \phi) = T_0 + T_1 \cos\theta \tag{2}$$

where both T_0 and T_1 are unknown. The quantity T_0 is the mean temperature of the planet, while the quantity $2T_1$ is the day-to-night time temperature difference we are trying to find.

Question 1: Sketch the geometry of the problem.

Question 2: Write down Laplace's equation for this problem (expanded in the spherical coordinate system), and the associated boundary conditions at $r = aR$ and $r = R$. Justify why this 3D problem can in fact be reduced to a 2D problem. Use separation of variables to express the problem as 2 coupled ODEs.

Question 3: Solve the angular equation (you may find textbook chapter 7.10 quite useful for that; however, note that the textbook uses a different notation for the two angular variables).

Question 4: Solve the radial equation (you may find textbook chapter 7.10 quite useful for that too).

Question 5: Fing the general solution to the problem. Hint: singular solutions in r must be kept - why?

Question 6: Apply the boundary conditions (1) and (2) to find the specific solution to our problem.

Question 7: To find what T_0 and T_1 are in terms of known quantities such as f_0 and $F_\star$, we must do a heat budget balance. On the day side, the heat budget is

$$F_\star + 2\pi(aR)^2 f_0 = 2\pi R^2 \int_0^{\pi/2} \sigma T^4(R, \theta) \sin\theta \mathrm{d}\theta \tag{3}$$

The terms on the left-hand-side are the input heat fluxes into the system, i.e. the total flux coming from the central star, and half the radioactive heat flux from the interior (integrated over the $r = aR$ hemisphere pointing towards the star). The term on the right-hand side is the heat lost through black body radiation. The heat flux of a black body is σT^4 at every point in the surface. The integral is taken over the surface hemisphere pointing towards the star (i.e. $\theta \in [0, \pi/2]$).

On the night side, the heat budget is similar but without the contribution from the central star. Write down a similar condition on the night-side. Be careful about the bounds of the integral term.

Question 8: Assume that $T_1 \ll T_0$ and show that

$$\int_0^{\pi/2} \sigma T^4 \sin\theta \mathrm{d}\theta \simeq \sigma T_0^4 \left(1 + 2\frac{T_1}{T_0}\right) \tag{4}$$

Do the same for the condition on the night-side, and deduce the values of T_0 and T_1 in terms of the known quantities in the system (geometry of the planet, input stellar and radioctive fluxes, etc).

Question 9: (Bonus question, if you have some physics background. No extra points, but good exercise!). Discuss the solution physically.