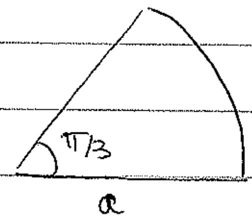
MIDTERM 2011 ANSWERS

Problem 1

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \quad \text{in a pie-shaped membrane}$$



$u = 0$ on the contour

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right]$$

let $u(r,\theta,t) = A(r,\theta)\,B(t)$

then

$$A B_{tt} = c^2 \left[B \nabla^2 A \right]$$

$$\Rightarrow \frac{B_{tt}}{B} = c^2 \frac{\nabla^2 A}{A} = -\omega^2$$

so $B_{tt} = -\omega^2 B$

and $\nabla^2 A = -\frac{\omega^2}{c^2} A$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A}{\partial \theta^2} = -\frac{\omega^2}{c^2} A$$

$$\Rightarrow r\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right) + \frac{r^2\omega^2}{c^2} A = -\frac{\partial^2 A}{\partial \theta^2}$$

Let $A(r,\theta) = a(r)\,b(\theta)$ then

$$\begin{cases} r\frac{\partial}{\partial r}\left(r\frac{\partial a}{\partial r}\right) + \frac{r^2\omega^2}{c^2} a = \lambda^2 a \\ \frac{\partial^2 b}{\partial \theta^2} = -\lambda^2 b \end{cases}$$

The last equation implies

$b(\theta) = \alpha\cos(\lambda\theta) + \beta\sin(\lambda\theta)$

$b(0) = b\left(\frac{\pi}{3}\right) = 0 \Rightarrow \alpha = 0$ and $\frac{\pi}{3}\lambda = n\pi$ so $\lambda_n = 3n$

$\Rightarrow b_n(\theta) = \sin(3n\theta) \qquad n \geq 1 \qquad$ ($n=0$ doesn't work).

$$\Rightarrow \quad r^2 \frac{d^2 a}{dr^2} + r\frac{da}{dr} + \frac{r^2\omega^2}{c^2} a = (3n)^2 a$$

$\Rightarrow$ let $x = \frac{\omega}{c} r$ so

$$x^2 \frac{d^2 a}{dx^2} + x\frac{da}{dx} + \left(x^2 - (3n)^2\right) a = 0$$

↳ this is the equation for the Bessel functions of order $3n$: so

$$a_n(x) = c_1 J_{3n}(x) + c_2 Y_{3n}(x)$$

↑ singular at $x=0$, so ignore

$$a_n(r) = J_{3n}\left(\frac{\omega}{c} r\right)$$

In order to have $a_n(a) = 0 \Rightarrow \frac{\omega}{c} a = z_{3n,m}$

so we have the eigenfrequencies

$$\boxed{\omega_{n,m} = z_{3n,m} \cdot \frac{c}{a}}$$

$z_{3n,m}$ = mth zero of the J_{3n} function

To find out which are the lowest eigenvalues, let's look at the table.

zeros of	1st	2nd	3rd	4th
J_3	6.38 (1)	9.76 (2)	13.01 (4)	16.22
J_6	9.93 (3)	13.58	17.00	[illegible]
J_9	13.35 (5)	17.24		

Problem 2

(1) Laplace's equation in spherical coordinates:

$$\begin{cases} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} = 0 \\ T(R,\theta,\phi) = T_0 + T_1 \cos\theta \\ -k \left.\frac{\partial T}{\partial r}\right|_{r=aR} = F_0 . \end{cases}$$

The boundary conditions do not depend on ϕ variable
→ the solution doesn't either so
$T = T(r,\theta)$ and the equation becomes

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) = 0$$

(2) let $T(r,\theta) = A(r)\,B(\theta)$
→

$$\begin{cases} \frac{d}{dr}\left(r^2 \frac{dA}{dr}\right) = \alpha A \\ \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dB}{d\theta}\right) = -\alpha B \end{cases}$$

The latter can be rewritten in terms of $\mu = \cos\theta$
with

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{dB}{d\mu}\right) = -\alpha B .$$

(3) Solutions of the μ-equation are Legendre Polynomials $P_n(\mu)$, with

$$\alpha = n(n+1). \qquad \text{(see 7.10.16 of textbook)}$$

(4) $$\frac{d}{dr}\left(r^2 \frac{dA_n}{dr}\right) = n(n+1) A_n$$

$\Rightarrow$ Try solutions as $A(r) = r^\alpha$ then we get

$$\alpha(\alpha+1) = n(n+1)$$

$$\alpha^2 + \alpha - n(n+1) = 0$$

2 solutions $\rightarrow \begin{cases} \alpha = n \\ \alpha = -n-1 \end{cases}$ so $A_n(r) = a_n r^n + b_n r^{-n-1}$

(5) Note that if $n=0$, we in fact have

$\begin{cases} P_0(\mu) \text{ as the angular solution } (P_0(\mu)=1) \\ A_0(r) = a_0 + \frac{b_0}{r} \end{cases}$

$\Rightarrow$ The general solution can be written as

$$T(r,\theta) = a_0 + \frac{b_0}{r} + \sum_{n=1}^{\infty} P_n(\cos\theta)\left[a_n r^n + b_n r^{-n-1}\right]$$

(6) Boundary conditions:

at $r=R$:

$$T_0 + T_1\cos\theta = a_0 + \frac{b_0}{R} + \sum_{n=1}^{\infty} P_n(\cos\theta)\left[a_n R^n + b_n R^{-n-1}\right]$$

However, $P_1(\cos\theta) = \cos\theta \Rightarrow$ we only need the $n=1$ term since the BCs is the sum of a P_0 and a P_1 function.

$$\Rightarrow \begin{cases} T_0 = a_0 + \dfrac{b_0}{R} \\ T_1 = a_1 R + b_1 R^{-2} \end{cases} \quad \text{and} \quad \begin{array}{c} 0 = a_n R^n + b_n R^{-n-1} \\ \forall n \geq 2 \end{array}$$

at $r = aR$:

$$F_0 = -k \left.\frac{\partial T}{\partial r}\right|_{r=aR}$$

$$= -k \left[-\frac{b_0}{a^2R^2} + \sum_{n=1}^{\infty}{}' P_n(\cos\theta) \left[n a_n (aR)^{n-1} - b_n(n+1)(aR)^{-n-2} \right] \right]$$

$$\Rightarrow \begin{cases} F_0 = \dfrac{k b_0}{a^2R^2} \\ 0 = n a_n (aR)^{n-1} - b_n (n+1)(aR)^{-n-2} \quad \forall n \geq 1. \end{cases}$$

Combining the two sets of bcs implies:

- $a_n = b_n = 0 \quad \forall n \geq 2$
- $b_0 = \dfrac{a^2R^2}{k} F_0 \quad \Rightarrow \quad a_0 = T_0 - \dfrac{a^2R}{k} F_0$
- $\begin{cases} T_1 = a_1 R + \dfrac{b_1}{R^2} \\ a_1 = 2b_1 (aR)^{-3} \end{cases} \rightarrow T_1 = \dfrac{2b_1}{a^3R^2} + \dfrac{b_1}{R^2}$

$$\Rightarrow \quad b_1 = \frac{T_1 R^2}{1 + \frac{2}{a^3}}$$

So

$$T(r,\theta) = T_0 - \frac{a^2R}{k} F_0 + \frac{a^2R^2}{k} \frac{F_0}{r}$$

$$+ \cos\theta \left\{ \frac{2}{a^3R^3} r + \frac{1}{r^2} \right\} \frac{T_1 R^2}{1 + \frac{2}{a^3}}$$

$$T(r,\theta) = T_0 + \frac{a^2R^2}{k}F_0\left(\frac{1}{r} - \frac{1}{R}\right)$$

$$+ \cos\theta\left\{\frac{2}{a^3}\left(\frac{r}{R}\right) + \left(\frac{R^2}{r^2}\right)\right\}\frac{T_1}{1+\frac{2}{a^3}}$$

(7) On the day-side

$$f_* + 2\pi(aR)^2 F_0 = 2\pi\int_0^{\pi/2} \sigma T^4 R^2 \sin\theta\, d\theta \qquad (a)$$

On the night side

$$2\pi(aR)^2 F_0 = 2\pi\int_{\frac{\pi}{2}}^{\pi} \sigma T^4 R^2 \sin\theta\, d\theta \qquad (b)$$

$$\int_0^{\frac{\pi}{2}} \sigma(T_0 + T_1\cos\theta)^4 \sin\theta\, d\theta \simeq \int_0^{\frac{\pi}{2}} \sigma T_0^4\left(1 + 4\frac{T_1}{T_0}\cos\theta\right)\sin\theta\, d\theta$$

$$\cong \sigma T_0^4\left(1 + 2\frac{T_1}{T_0}\right)$$

$$\int_{\frac{\pi}{2}}^{\pi} \sigma(T_0 + T_1\cos\theta)^4 \sin\theta\, d\theta \simeq \int_{\pi/2}^{\pi} \sigma T_0^4\left(1 + \frac{4T_1}{T_0}\cos\theta\right)\sin\theta\, d\theta$$

$$= \sigma T_0^4\left(1 - 2\frac{T_1}{T_0}\right)$$

So we have

$(a-b)$: $$f_* = 2\pi R^2 \sigma T_0^4 \cdot 4\frac{T_1}{T_0}$$

$(a+b)$ $$f_* + 4\pi(aR)^2 F_0 = 2\pi R^2 \cdot 2\sigma T_0^4$$

$$= 4\pi R^2 \sigma T_0^4$$

This implies that

$$T_0 = \left(\frac{F_* + 4\pi (a^2 R^2) F_0}{4\pi R^2 \sigma} \right)^{1/4}$$

and

$$T_1 = \frac{F_*}{8\pi R^2 \sigma T_0^3}$$