FINAL 2011

Each problem is graded out of 100/100

Problem 1

20 (1) If

$$\nabla^2 u_1 = 0 \qquad u_1(0,y) = a_1(y) \quad \text{all other sides } 0$$
$$\nabla^2 u_2 = 0 \qquad u_2(L,y) = a_2(y) \quad \text{------}$$
$$\nabla^2 u_3 = 0 \qquad u_3(x,0) = a_3(x) \quad \text{------}$$
$$\nabla^2 u_4 = 0 \qquad u_4(x,H) = a_4(x) \quad \text{----}$$

$\rightarrow \quad U = U_1 + U_2 + U_3 + U_4$

satisfy $\quad \nabla^2 u = 0$ with BC given by (1).

60 (2) Solution for u_1

Solutions on $y=0$ and $y=H$ sides homogeneous $\Rightarrow$
use this side for sine-expansion:

$$u_1(x,y) = \sum_{n=1}^{\infty} A_n(x) \sin\left(\frac{n\pi y}{H}\right)$$

with $\quad \dfrac{d^2 A_n}{dx^2} - \dfrac{n^2\pi^2}{H^2} A_n = 0 \quad$ to satisfy $\nabla^2 u_1 = 0$

$$\rightarrow \quad A_n(x) = \alpha_n \exp\left(-\frac{n\pi x}{H}\right) + \beta_n \exp\left(\frac{n\pi x}{H}\right)$$

To satisfy $\quad u_1(L,y) = 0 \quad \Rightarrow \quad \rightsquigarrow$

$$A_n(x) \propto \sinh\left(\frac{n\pi(x-L)}{H}\right)$$

So $\quad u_1(x,y) = \displaystyle\sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi y}{H}\right) \sinh\left(\frac{n\pi(x-L)}{H}\right)$

$\Rightarrow$

applying BC $\Rightarrow$

$$U_1(0,y) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi y}{H}\right) \sinh\left(-\frac{n\pi L}{H}\right) = a_1(y)$$

$$\Rightarrow \quad \alpha_n \sinh\left(-\frac{n\pi L}{H}\right) = \frac{2}{H}\int_0^H a_1(y) \sin\left(\frac{n\pi y}{H}\right) dy$$

$$\Rightarrow \quad U_1(x,y) = \sum_{n=1}^{\infty} -\frac{2}{H\sinh\left(\frac{n\pi L}{H}\right)} \int_0^H a_1(y') \sin\left(\frac{n\pi y'}{H}\right) dy' \cdot \sin\left(\frac{n\pi y}{H}\right) \sinh\left(\frac{n\pi (x-L)}{H}\right)$$

2. (3) <u>Solution for U_2, U_3, U_4</u>

U_2: This time $A_n \propto \sinh\left(\frac{n\pi x}{H}\right)$

so $$U_2(x,y) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi y}{H}\right) \sinh\left(\frac{n\pi x}{H}\right)$$

$\Rightarrow$ applying BC

$$U_2(L,y) = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n\pi y}{H}\right) \sinh\left(\frac{n\pi L}{H}\right) = a_2(y)$$

$$\Rightarrow \quad \alpha_n = \frac{2}{H\sinh\left(\frac{n\pi L}{H}\right)} \int_0^H a_2(y) \sin\left(\frac{n\pi y}{H}\right) dy$$

$$U_2(x,y) = \sum_{n=1}^{\infty} \frac{2}{H\sinh\left(\frac{n\pi L}{H}\right)} \int_0^H a_2(y) \sin\left(\frac{n\pi y'}{H}\right) dy' \cdot \sin\left(\frac{n\pi y}{H}\right) \sinh\left(\frac{n\pi x}{H}\right)$$

By switching $x \leftrightarrow y$, $H \leftrightarrow L$, we get U_3 & U_4:

$$U_3(x,y) = \sum_{n=1}^{\infty} -\frac{2}{L\sinh\left(\frac{n\pi H}{L}\right)} \int_0^L a_3(x') \sin\left(\frac{n\pi x'}{L}\right) dx' \cdot \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi (y-H)}{L}\right)$$

$$U_4(x,y) = \sum_{n=1}^{\infty} \frac{2}{L\sinh\left(\frac{n\pi H}{L}\right)} \int_0^L a_4(x) \sin\left(\frac{n\pi x'}{L}\right) dx' \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right)$$

Problem 2

(1) $bc \frac{\partial a}{\partial t} = D\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 ac \frac{\partial b}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \, ab \frac{\partial c}{\partial \theta}\right) - \frac{abc}{r^2 \sin^2\theta}\right]$

$\Rightarrow \frac{1}{D}\frac{1}{a}\frac{\partial a}{\partial t} = \frac{1}{b}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial b}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{1}{c}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial c}{\partial \theta}\right) - \frac{1}{r^2 \sin^2\theta}$

$\Rightarrow \boxed{\frac{da}{dt} = -aD\lambda}$ λ = arbitrary coupling constant #1

$\Rightarrow -r^2\lambda = \frac{1}{b}\frac{\partial}{\partial r}\left(r^2 \frac{\partial b}{\partial r}\right) + \frac{1}{\sin\theta}\frac{1}{c}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial c}{\partial \theta}\right) - \frac{1}{\sin^2\theta}$

$\Rightarrow -\frac{1}{b}\frac{\partial}{\partial r}\left(r^2 \frac{\partial b}{\partial r}\right) - r^2\lambda = -K = \frac{1}{\sin\theta}\frac{1}{c}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial c}{\partial \theta}\right) - \frac{1}{\sin^2\theta}$

arabitrary coupling constant #2.

$\Rightarrow \boxed{\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial c}{\partial \theta}\right) - \frac{c}{\sin^2\theta} + cK = 0}$

$\boxed{\frac{\partial}{\partial r}\left(r^2 \frac{\partial b}{\partial r}\right) + r^2\lambda b = Kb}$ 20

(2) Using change of variable $x = \cos\theta$, 20

$\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial c}{\partial x}\right) - \frac{c}{(1-x^2)} + cK = 0$

$\Rightarrow$ This is the eq. for an Associated Legendre function $P_n^1(x)$ provided $K_n = n(n+1)$

(3) The radial eq becomes 20

$r^2 \frac{\partial^2 b}{\partial r^2} + 2r\frac{\partial b}{\partial r} + r^2\lambda b - n(n+1)\, b = 0$

$\rightarrow$ with change of variable $z = \sqrt{\lambda}\, r$ we get

$$z^2 \frac{d^2 b}{dz^2} + 2z\frac{db}{dz} + z^2 b - n(n+1)b = 0$$

$$\Rightarrow \quad b = j_n(z)$$

$$= j_n(\sqrt{\lambda}\, r)$$

The eigenvalue λ is chosen so that $b(R) = 0$ (by boundary condition) so

$$j_n(\sqrt{\lambda}\, R) = 0 \quad \Rightarrow \quad \sqrt{\lambda}\, R = \text{one of the zeros of } j_n \text{, call } z_{mn}$$

$$\Rightarrow \quad \lambda_{mn} = \left(\frac{z_{mn}}{R}\right)^2.$$

5 (4) To show this, note that

$$\int_0^R b_{mn}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial b_{m'n}}{\partial r}\right) + r^2 \lambda_{m'n} b_{m'n} - n(n+1) b_{m'n}\right] dr = 0$$

$$\Rightarrow \quad \int_0^R \left\{-r^2 \frac{\partial b_{mn}}{\partial r}\frac{\partial b_{m'n}}{\partial r} + \left[r^2 \lambda_{m'n} - n(n+1)\right] b_{mn} b_{m'n}\right\} dr$$

$$+ \left[b_{mn}\, r^2 \frac{\partial b_{m'n}}{\partial r}\right]_0^R = 0$$

but $b_{mn}(R) = 0$ by definition

& $r^2 \frac{\partial b_{m'n}}{\partial r} = 0$ too by regularity since b is bounded.

$$* \quad \Rightarrow \quad \int_0^R r^2 \lambda_{m'n} b_{mn} b_{m'n}\, dr = \int_0^R \left[r^2 \frac{\partial b_{mn}}{\partial r}\frac{\partial b_{m'n}}{\partial r} - n(n+1) b_{mn} b_{m'n}\right] dr$$

From a similar equation $\int_0^R b_{m'n}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial b_{mn}}{\partial r}\right) + \cdots\right] dr$

we get $\int_0^R r^2 \lambda_{mn} b_{mn} b_{m'n}\, dr = \int_0^R \left[r^2 \frac{\partial b_{mn}}{\partial r}\frac{\partial b_{m'n}}{\partial r} - n(n+1) b_{mn} b_{m'n}\right] dr$

$$\Rightarrow \quad \int_0^R r^2 \lambda_{mn} b_{mn} b_{m'n}\, dr = \int_0^R r^2 \lambda_{m'n} b_{mn} b_{m'n}\, dr$$

but $\lambda_{mn} \neq \lambda_{m'n}$ so if $m \neq m'$

$$\int_0^R r^2 b_{mn} b_{m'n}\, dr = 0 \text{ if } m \neq m'.$$

20 (5) Temporal eq $\Rightarrow$ $a(t) = e^{-\lambda D t}$

$$\Rightarrow$$

$$B(r,\theta,t) = \sum_{m,n=} d_{mn} P_n^1(\cos\theta)\, j_n\left(\frac{z_{mn}}{R} r\right) e^{-\left(\frac{z_{mn}}{R}\right)^2 Dt}$$

5 (6) $P_n^1(x) = (1-x^2)^{1/2} \dfrac{dP_n}{dx}$

$P_0(x) = 1 \Rightarrow P_0^1(x) = 0$

$P_1(x) = x \Rightarrow P_1^1(x) = \sqrt{1-x^2}$

$P_1^1(\cos\theta) = \sqrt{1-\cos^2\theta} = \sin\theta$

that's the angular dependence of IC

$$\Rightarrow B(r,\theta,t) = \sum_m d_m \sin\theta\, j_1\left(\frac{z_{m1}}{R} r\right) e^{-\left(\frac{z_{m1}}{R}\right)^2 Dt}$$

10 (7) To apply the BCs, note that

$$B(r,\theta,0) = B_0(r)\sin\theta = \sum_m d_m j_1\left(\frac{z_{m1}}{R} r\right)\sin\theta$$

$$\Rightarrow B_0(r) = \sum_m d_m j_1\left(\frac{z_{m1}}{R} r\right) = \sum_m d_m b_{m1}(r)$$

$$\Rightarrow d_m = \frac{\int_0^R r^2 B_0(r) j_1\left(\frac{z_{m1}}{R} r\right) dr}{\int_0^R r^2 \left[j_1\left(\frac{z_{m1}}{R} r\right)\right]^2 dr} = \frac{\int_0^R r^2 B_0(r)\, b_{m1}(r)\, dr}{\int_0^R r^2 b_{m1}^2(r)\, dr}.$$

Problem 3

$$u_{xx} - u_{xy} - 6u_{yy} = 0 \qquad a=1 \quad b=-\frac{1}{2} \quad c=-6$$

(1) $\delta = \frac{1}{4} + 6 = \frac{25}{4} > 0 \rightarrow$ hyperbolic 10.

(2) $$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2-ac}}{a} = \frac{-\frac{1}{2} \pm \sqrt{\frac{25}{4}}}{1} = -\frac{1}{2} \pm \frac{5}{2} = \begin{cases} 2 \\ -3 \end{cases}$$

$$\rightarrow \begin{cases} y = 2x + \xi \\ y = -3x + \eta \end{cases} \qquad \begin{cases} \boxed{\xi = y - 2x} & 10. \\ \boxed{\eta = y + 3x} & 10 \end{cases}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = -2\frac{\partial}{\partial \xi} + 3\frac{\partial}{\partial \eta}$$

Similarly $\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}$

$$\Rightarrow u_{xx} - u_{xy} - 6u_{yy} = 0$$

$$\Rightarrow \left(-2\frac{\partial}{\partial \xi} + 3\frac{\partial}{\partial \eta}\right)^2 u - \left(-2\frac{\partial}{\partial \xi} + 3\frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)u - 6\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)^2 u = 0$$

$$\Rightarrow \cancel{4u_{\xi\xi}} - 12u_{\xi\eta} + \cancel{9u_{\eta\eta}} + \cancel{2u_{\xi\xi}} - u_{\xi\eta} - \cancel{3u_{\eta\eta}} - \cancel{6u_{\xi\xi}} - 12u_{\xi\eta} - \cancel{6u_{\eta\eta}} = 0$$

$$\Rightarrow -25u_{\xi\eta} = 0 \Rightarrow u_{\xi\eta} = 0$$ as required. 20.

(3) $$u_{\xi\eta} = 0 \Rightarrow u(\xi,\eta) = F(\xi) + G(\eta)$$

$$\Rightarrow u(x,y) = F(y-2x) + G(y+3x)$$ 20.

(4) $\begin{cases} u(x,0) = f(x) \\ u_y(x,0) = g(x) \end{cases}$

$$\Rightarrow \begin{cases} F(-2x) + G(3x) = f(x) & 5 \\ F'(-2x) + G'(3x) = g(x) & 5 \end{cases}$$

$$\Rightarrow \begin{cases} F(-2x) + G(3x) = f(x) \\ -\frac{1}{2}\left[F(-2x) - F(0)\right] + \frac{1}{3}\left[G(3x) - G(0)\right] = \int_0^x g(x')dx' \end{cases}$$

$$\Rightarrow \begin{cases} F(-2x) + G(3x) = f(x) \\ -\frac{1}{2}F(-2x) + \frac{1}{3}G(3X) = \int_0^x g(x')dx' - \frac{F(0)}{2} + \frac{G(0)}{3} \end{cases}$$

$$\Rightarrow \frac{5}{3}G(3x) = f(x) + 2\int_0^x g(x')dx') - F(0) + \frac{2G(0)}{3}.$$

$$\frac{5}{3}G(x) = f\left(\frac{x}{3}\right) + 2\int_0^{\frac{x}{3}} g(x')\,dx' - F(0) + \frac{2}{3}G(0)$$

$$\frac{5}{2}F(-2x) = f(x) - 3\int_0^x g(x')dx' + \frac{3}{2}F(0) - G(0)$$

$$\frac{5}{2}F(x) = f\left(-\frac{x}{2}\right) - 3\int_0^{-\frac{x}{2}} g(x')dx' + \frac{3}{2}F(0) - G(0)$$

$$\Rightarrow u(x,y) = \frac{2}{5}\left[f\left(-\frac{y-2x}{2}\right) - 3\int_0^{\frac{2x-y}{2}} g(x')dx' + \frac{3}{2}F(0) - G(0)\right]$$
$$+ \frac{3}{5}\left[f\left(\frac{y+3x}{3}\right) + 2\int_0^{\frac{y+3x}{3}} g(x')dx') - F(0) + \frac{2}{3}G(0)\right]$$

$$u(x,y) = \frac{2}{5} f\left(\frac{2x-y}{2}\right) + \frac{3}{5} f\left(\frac{y+3x}{3}\right)$$

$$+ \frac{6}{5} \int_{\frac{2x-y}{2}}^{\frac{y+3x}{3}} g(x')\,dx' \quad + \cancel{\frac{3}{5} f(0)} - \cancel{\frac{3}{5} f(0)} - \cancel{\frac{2}{5} G(0)} + \cancel{\frac{2}{5} G(0)}$$

$$= \frac{2}{5} f\left(x - \frac{y}{2}\right) + \frac{3}{5} f\left(\frac{y}{3} + x\right) + \frac{6}{5} \int_{x-\frac{y}{2}}^{x+\frac{y}{3}} g(x')\,dx'$$

20

Problem 4

$$u_t + (uv(u))_x = 0$$

$v(u) = 1-u$ if $0 < u < 1$, 0 otherwise

$$u(x,0) = \begin{cases} a & \text{if } x < 0 \\ a + (b-a)x & \text{if } x \in [0,1] \\ b & \text{if } x > 1 \end{cases}$$

(1) $u_t + (u(1-u))_x = 0$

$\Rightarrow u_t + (u - u^2)_x = 0$

$\Rightarrow u_t + (1-2u)u_x = 0$

$$C^{(s)}: \left.\begin{cases} \frac{dt}{d\tau} = 1 & \longrightarrow \quad t = \tau \\ \frac{dx}{d\tau} = (1-2u) & \longrightarrow \quad x = (1-2u_0(s))\tau + s \\ \frac{du}{d\tau} = 0 & \longrightarrow \quad u = u_0(s) \end{cases}\right\} 2c.$$

Characteristics satisfy $\boxed{t = \frac{x-s}{1-2u_0(s)}}$.

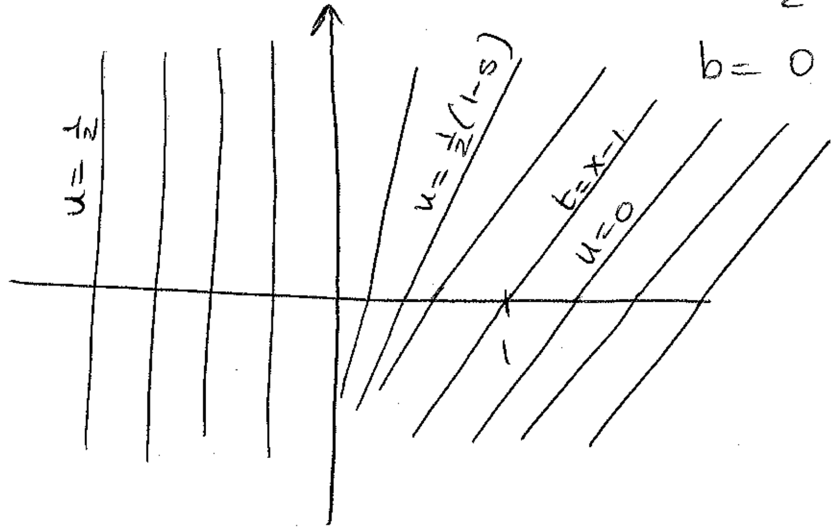
On the characteristics

$s < 0$: $t = \frac{x-s}{1-2a}$, $u = a$

$0 < s < 1$: $t = \frac{x-s}{1-2(a+(b-a)s)}$ $u = a + (b-a)s$

$s > 1$: $t = \frac{x-s}{1-2b}$, $u = b$.

(2)



$a=\frac{1}{2} \Rightarrow 1-2a=0$

$b=0 \Rightarrow 1-2b=1$

10 for chars.
5 for transition

since

On
$$\begin{cases} s\le 0 & x=s\ , \quad u=\frac{1}{2} \qquad 5 \\ s\ge 1 & t=\frac{x-s}{1}=x-s\ ; \quad u=0 \qquad 5 \\ s\in[0,1] & t=\dfrac{x-s}{1-2\left(\frac{1}{2}(1-s)\right)}=\dfrac{x-s}{1-(1-s)}=\dfrac{x-s}{s} \end{cases}$$

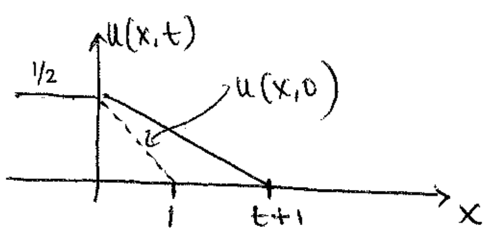
In last case

$$\Rightarrow st=x-s \Rightarrow s=\frac{x}{t+1}$$

$$u=\frac{1}{2}(1-s)=\frac{1}{2}\left(1-\frac{x}{t+1}\right) \quad 5.$$

(3) $\Rightarrow$ The full solution is

$$\begin{cases} u(x,t)=\frac{1}{2} & \text{for} \quad x\le 0 \\ u(x,t)=\frac{1}{2}\left(1-\frac{x}{t+1}\right) & 0<x\le t+1 \\ u(x,t)=0 & x\ge t+1 \end{cases}$$



10 (5 each plot).

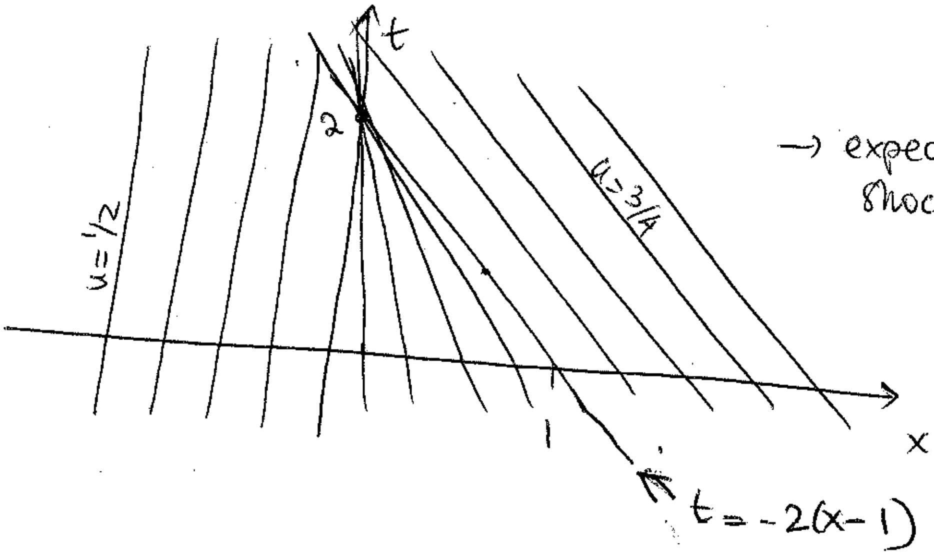
(4) If $a=\frac{1}{2}$ $\quad b=\frac{3}{4} \rightarrow 1-2b=-\frac{1}{2}$

$$\begin{cases} s<0 & x=s, \quad u=\frac{1}{2} \\ s>0 & t=\dfrac{x-s}{-\frac{1}{2}}=-2(x-s) \end{cases}$$

$$s\in[0,1] \qquad t=\frac{x-s}{1-2\left(\frac{1}{2}+\frac{1}{4}s\right)}=\frac{x-s}{-\frac{1}{2}s}=-\frac{2(x-s)}{s}$$

10 If $x=0$, $t=4$ $\Rightarrow$ all characteristics cross @ $(0,2)$

(5)



10 $\rightarrow$ expect a compression shock starting at $(0,2)$

(6) for $t<2$:

In the intermediate region

$$t=-\frac{2(x-s)}{s} \Rightarrow -st=2(x-s)=2x-2s$$

$$\Rightarrow (2-t)s=2x$$

$$\Rightarrow s=\frac{2x}{2-t}$$

so $u=\frac{1}{2}+\frac{1}{4}s=\frac{1}{2}+\frac{1}{4}\frac{2x}{2-t}$

10
$$\Rightarrow \begin{cases} u=\frac{1}{2} & x<0 \\ u=\frac{1}{2}+\dfrac{x}{2(2-t)} & 0\le x\le\dfrac{2-t}{2} \\ u=\frac{3}{4} & x\ge\dfrac{2-t}{2}. \end{cases}$$

for $t > 2$, a shock develops between the region with $u_- = \frac{1}{2}$ and $u_+ = \frac{3}{4}$.

$$\frac{d\gamma}{dt} = \frac{F(u_+) - F(u_-)}{u_+ - u_-} = \frac{u_+ V(u_+) - u_- V(u_-)}{u_+ - u_-} = \frac{\frac{3}{4}\cdot\frac{1}{4} - \frac{1}{2}\cdot\frac{1}{2}}{\frac{3}{4} - \frac{1}{2}}$$

$$= \frac{\frac{3}{16} - \frac{1}{4}}{\frac{1}{4}} = -\frac{1}{4} \implies \gamma = -\frac{1}{4}t + \text{const}.$$

We fit the constant to have $\gamma = 0$ @ $t = 2$

$$\implies 0 = -\frac{1}{4}2 + \text{constant}$$

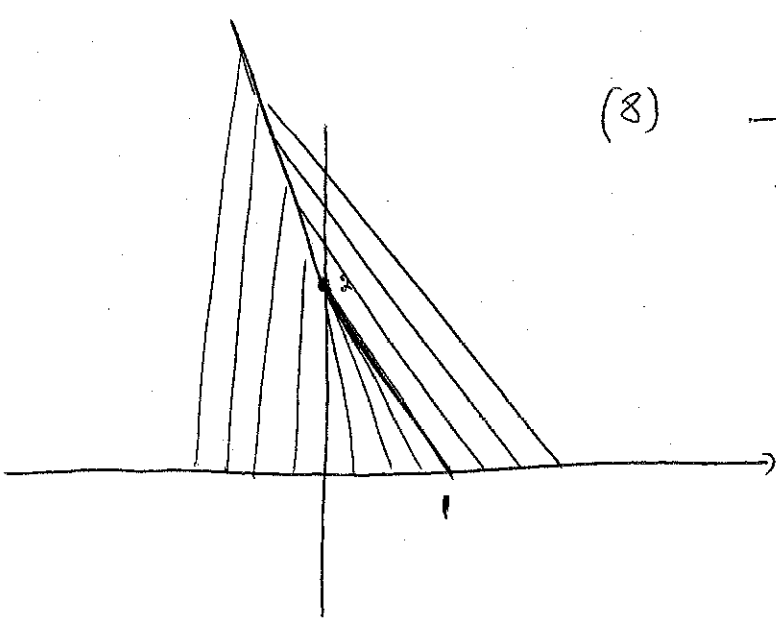
$$\implies \text{constant} = \frac{1}{2}$$

so $\implies \gamma = \frac{1}{2} - \frac{1}{4}t \implies t = -4\left(\gamma - \frac{1}{2}\right)$

$$= 2 - 4\gamma$$

$$\rightarrow \begin{cases} u(x,t) = \frac{1}{2} & x < \frac{1}{2} - \frac{1}{4}t \\ u(x,t) = \frac{3}{4} & x > \frac{1}{2} - \frac{1}{4}t \end{cases}$$

(7)



(8)