AMS 212A Final 2010

There are 4 problems on this final. Each is worth 30 points, so that answering 3 problems perfectly is sufficient for an A. Points above 100 will count as extra credit on your midterm grade.

You need to justify all your answers. Answers without justifications will be counted as wrong. If you have any hesitation about the question, if you think there may be a problem with it or if you just need clarification, DON'T HESITATE TO ASK!

Calculators are not allowed.

Problem 1: Expansion shock as the limit of a well-posed problem.

This problem illustrates how the solution for an expansion shock can actually be re-derived as the limit of a well-posed problem. Consider the PDE

$$u_t + uu_x = 0 \tag{1}$$
$$u(x,0) = u_0(x) = \tanh\left(\frac{x}{\epsilon}\right) \text{ for all } x$$

where we assume that $\epsilon > 0$.

- Show that this problem has a solution by calculating the expression for the characteristics.
- What is the limit of the slope of the characteristics as $s \to -\infty$? as $s \to +\infty$? Based on your results, draw the characteristics on the (x,t) plane.
- Find the implicit expression for the solution for all times $t \geq 0$.
- Draw $u_0(s)$ for a large value and a small value of ϵ. What is $\lim_{\epsilon \to 0} u_0(s)$? From now on, consider ϵ to be very small.
- What is the limit of $u_0(s)$ on characteristics with $|s| \gg \epsilon$? Deduce the limit of $u(x,t)$ on these characteristics. HINT: don't forget there are two cases to consider, $s > 0$ and $s < 0$.
- What is the limit of $u_0(s)$ on characteristics with $|s| \ll \epsilon$? Find $x(s,\tau)$, and solve for s to deduce the limit of $u(x,t)$ on those characteristics. NOTE: you may assume that $t \gg \epsilon$.
- We can finally compare this expression with the solution we would have obtained with the following initial condition instead:

$$u_0(s) = -1 \text{ for } s \leq 0$$
$$u_0(s) = 1 \text{ for } s > 0$$

Solve for the resulting solution, including the expansion shock, and write the slution for $u(x,t)$ in all three relevant regions. Does the result recover what you found in the previous questions with $\epsilon \to 0$?

Problem 2: Canonical forms

Consider the equation and initial conditions:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \\ u(x,0) &= f(x) \\ u_t(x,0) &= g(x) \end{aligned} \tag{2}$$

Solve the problem to show that it indeed recovers d'Alembert's solution (see Formula sheet) in the absence of forcing.

Problem 3:

In this problem we study a system related to the eigenmodes of a didgeridoo. A didgeridoo is (roughly speaking) a long wooden cylinder open at both ends. By blowing on one end, a sound is created.

Here we consider instead, for simplicity, a slightly different problem, where the ends of the didgeridoo are closed. A sound can then be made by gently tapping it, thus exciting sound waves in the air trapped within.

The instrument we're looking at is now modelled as a perfect cylinder, of length L and radius a. The equation describing the propagation of the sound waves within is

$$u_{tt} = c_s^2 \left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r} \right) + \frac{\partial^2 u}{\partial z^2} \right] \tag{3}$$

for $z \in (0, L)$, and $r \in (0, a)$. The velocity of the waves $u(r, z, t)$ is zero on all walls, and is always finite.

(a) How would you express the previous boundary and regularity conditions mathematically?

(b) Use separation of variables to reduce the system to 3 coupled ODEs. HINT: you will have a choice to make in which terms to group with one another. The two ways of doing it are equivalent, so just pick one.

(c) Solve the problem in the z direction first, and apply boundary conditions to find the $z-$eigenmodes.

(d) Show that the $r-$direction problem can be recast into the following Bessel Equation:

$$x^2 u_{xx} + x u_x + x^2 u = 0$$

for a judicious choice of the variable x. Note that this seems to imply a condition on the eigenfrequencies, but you shouldn't be worried about it.

(e) This Bessel equation has two solutions, $J_0(x)$ and $Y_0(x)$. Note that $Y_0(x)$ is singular at $x = 0$. By applying the boundary conditions at $r = a$, find the radial eigenmodes of the system as well as the eigenfrequencies of the closed didgeridoo. You may use the notation that the m-th zero of the J_0 function is called z_m (by convention, z_1 is the first zero).

(f) What is the lowest possible eigenfrequency of this instrument (careful, it's a bit of a trick question)?

Problem 4:

Consider the 1D heat diffusion problem with non-constant diffusivity, in a very thin rod of length L. The rod is insulated on all sides and initially at a non-zero temperature. This problem is cast mathematically as

$$\begin{aligned} T_t &= ((1+x)^2 T_x)_x \\ T(0,t) &= T(L,t) = 0 \\ T(x,0) &= T_0(x) \end{aligned} \tag{4}$$

(a) First recast this problem into a more easily solvable one by using the new spatial variable $y = 1 + x$. What are the new equation and the new boundary conditions in this case?

(b) Consider the Sturm-Liouville problem

$$(y^2 u_y)_y + \lambda u = 0 \text{ for } 1 < y < a \text{ with } u(1) = u(a) = 0$$

Using the Rayleigh Quotient, show that all the eigenvalues have to be positive.

(c) Show that (i.e. do not merely verify) the eigenfunctions of the Sturm-Liouville problem above are

$$v_n(y) = y^{-1/2} \sin(\alpha_n \ln(y))$$

where the constants α_n needs to be determined. Also find the eigenvalues (note that these are not the α_n).

(d) What orthogonality relation do the $v_n(y)$ functions satisfy?

(e) Go back to the original time-dependent problem, reformulated in y. Use separation of variables to find the formal solution (in terms of y first, and then of x). This solution contains a set of unknown constants, which you should then express as integrals over the initial conditions.