Final

IMPORTANT INSTRUCTIONS: All problems carry equal weight. Do not hand in more than 3 of the 6 problems; you must select at least one from 1,2,3 and one from 4,5,6. Note that all questions are similar in length and difficulty.

Extra credit: you may work out, scan and email me solutions (as pdf please) to the three additional problems for extra credit no later than midnight Callifornia time, on Wednesday 7th, 2005.

Problem 1: Fully nonlinear first order PDEs

The Eikonal equation from Geometrical Optics is an asymptotic approximation to the wave equation in the limit where the wavelength is much smaller than any spatial variation in the wave-speed. It is given by

$$u_x^2 + u_y^2 = n^2(x, y)$$

where $n(x, y)$ is the local refraction index of the medium through which the wave is propagating. The solution $u(x, y)$ is the wave amplitude; lines of constant u can be thought of as wave-fronts.

We consider the semi-infinite strip defined as $x \in [0, 1]$ and $y \geq 0$ (see Figure 1); the refraction index is defined as

$$n(x, y) = n_1 \text{ if } y \in [0, 1]$$
$$n(x, y) = n_2 \text{ if } y > 1$$

The following problem guides you through the determination of $u(x, y)$ solution of the Eikonal equation with the above refraction index.

Question 1: We consider the following initial condition for the problem:

$$u\left(x, \frac{x}{2}\right) = 1$$

On the diagram in Figure 1, draw the curve separating the $n(x, y) = n_1$ and $n(x, y) = n_2$ regions. On the same diagram, draw the initial condition curve.

Question 2: What are the characteristic equations $(x_\tau, y_\tau, u_\tau, p_\tau, q_\tau)$ and the initial conditions $(x_0(s), y_0(s), u_0(s), p_0(s), q_0(s))$ for the problem in region 1 ($y \in [0, 1]$)? (Note: here $p = u_x$ and $q = u_y$).

Question 3: Solve these equations for the characteristic coordinates (τ, s), and then invert the solution for (x, y) to show that

$$u(x, y) = 1 - \frac{n_1}{\sqrt{5}}(x - 2y) \text{ for } y \in [0, 1]$$

Draw lines of constant u for $y \leq 1$ on Figure 1 as dotted lines.

Question 4: Determine $u(x, 1)$, the value of the solution on the interface between the two regions. We now use this as an initial condition curve to find the solution for u in region 2 ($y > 1$). What are the new characteristic equations $(x_\tau, y_\tau, u_\tau, p_\tau, q_\tau)$ and the new "initial conditions" $(x_0(s), y_0(s), u_0(s), p_0(s), q_0(s))$ for the problem in region 2 ($y > 1$)? What is the condition on the values of n_1 and n_2 for $q_0(s)$ to be defined?

Question 5: Assume that q_0 is indeed defined, then solve the problem in region 2 to show that

$$u(x, y) = 1 - \frac{n_1}{\sqrt{5}}(x - 2) + \frac{y - 1}{\sqrt{5}}\sqrt{5n_2^2 - n_1^2}$$

Question 6: What are the equations for lines of constant u in region 2?

Question 7: Calculate $\sin\theta_1$ and $\sin\theta_2$, where θ_1 and θ_2 are the angles between lines of constant u and the horizontal (i.e. the x-axis) in region 1 and region 2 respectively. Verify that these satisfy Snell's Law:

$$n_2 \sin\theta_2 = n_1 \sin\theta_1$$

Problem 2: Shocks

Given the conservation law valid for $t \geq 0$:

$$u_t + uu_x = 0$$

with initial condition

$$\begin{aligned} u(x, 0) = \phi(x) &= 1 \text{ if } x < -1 \\ &= -1 \text{ if } x \in [-1, 0) \\ &= 1 \text{ if } x \in [0, 1) \\ &= -1 \text{ if } x \geq 1 \end{aligned}$$

Question 1: Plot $\phi(x)$.

Question 2: Determine the equation for the characteristics lines emanating from each of the 4 regions, and draw them with a pencil on Figure 2 (you may need to erase part of your diagram later).

Question 3: Identify the rarefaction region, as well as the two initial shock positions at $t = 0$.

Question 4: Find the solution $u(x, t)$ in the rarefaction region and sketch the corresponding characteristics on Figure 2 (with a pencil).

Question 5: Determine the equation for the two shock fronts $x_1 = \gamma_1(t)$ (located in the $x < 0$ domain) and $x_2 = \gamma_2(t)$ (located in the $x > 0$ domain) for $t \leq 1$. Draw the shock fronts on Figure 2. What happens at $t = 1$?

Question 6: Consider the shock front $\gamma_2(t)$. Show that for $t \in [1, T]$ (where T is to be determined)

$$\gamma_2(t) = 2\sqrt{t} - t$$

and by symmetry, or otherwise, determine $\gamma_1(t)$.

Question 7: With the help of the plot given in Figure 3, or otherwise, continue the sketch of the shock fonts on Figure 2. Show mathematically that the two shock fronts intercept at $T = 4$.

Question 8: By symmetry arguments, or otherwise, determine the equation for the single resulting shock front for $t > T$. Finish the diagram in Figure 2, making sure to indicate clearly (1) all the shock fronts (2) some relevant characteristic lines. In each region, indicate the value of the solution $u(x,t)$.

Problem 3: Canonical Forms

Consider the equation and associated conditions

$$\begin{aligned} u_{xx} - 2u_{xy} + 4e^y &= 0 \\ u(0,y) &= f(y) \\ u_x(0,y) &= g(y) \end{aligned}$$

Question 1: What is the type of this equation?

Question 2: By performing an adequate change of coordinates reduce the equation to its canonical form.

Question 3: Show that the solution to this problem is given by

$$u(x,y) = (2x+1)e^y - e^{y+2x} + f(y) + \frac{1}{2}\int_y^{y+2x} g(s)\mathrm{d}s$$

Note: simply plugging the solution into the equation will earn you no points.

Problem 4: 1-D wave equation on the semi-infinite line with non-homogeneous boundary conditions

This problem guides you towards finding a general solution to the following problem:

$$\begin{aligned} &u_{tt} - u_{xx} = 0 \text{ for } x > 0 \\ &u(0,t) = h(t) \text{ for } t \geq 0 \\ &u(x,0) = f(x) \text{ for } x > 0 \\ &u_t(x,0) = g(x) \text{ for } x > 0 \end{aligned}$$

Question 1: Why is it not possible to write d'Alembert's solution for this problem for all (x_0, t_0)? (Hint: distinguish between the cases $x_0 - t_0 > 0$ and $x_0 - t_0 \leq 0$. You may find it easier to explain your answer with two suitable diagrams illustrating d'Alembert's method in the (x,t) plane.). How would you interpret the distinction between the two regions in terms of causality?

Question 2: Write the solution to the above problem for $x_0 - t_0 > 0$.

To solve the problem in the region $x_0 - t_0 \leq 0$, we need to use the parallelogram identity illustrated in Figure 4 (you do not need to prove it):

$$u(B^+) + u(B^-) = u(A^+) + u(A^-)$$

where the points A^-, B^+, A^+ and B^- are four corners of a parallelogram where the sides are aligned with the characteristic lines (i.e. lines of constant $x - t$ and lines of constant $x + t$). Note: the notation $u(A)$ denotes $u(x_A, t_A)$ where (x_A, t_A) are the coordinates of the point A in the (x,t) plane.

Question 3: Consider the parallelogram which has one corner at the point (x_0, t_0) and then two of the remaining three corners on the $x-$axis and on the $t-$axis respectively. Draw it on a suitable diagram.

What are the coordinates of the 4 corners?

Question 4: Using the parallelogram identity, show that

$$u(x_0, t_0) = h(t_0 - x_0) - f(t_0 - x_0) + u(t_0, x_0)$$

Question 5: Deduce that

$$u(x_0, t_0) = h(t_0 - x_0) + \frac{1}{2}\left(f(t_0 + x_0) - f(t_0 - x_0)\right) + \frac{1}{2}\int_{t_0 - x_0}^{t_0 + x_0} g(s)\mathrm{d}s$$

Question 6: Solve the problem for $f(x) = 2x$, $g(x) = e^{4x}$ and $h(t) = t/(1-t)$, making sure to distinguish between the solutions in the two regions. Is the solution continuous?

Problem 5: Separation of variables

The following problem guides you to find solutions to the Beam Equation on a finite domain $x \in [0, L]$. The Beam Equation and associated conditions are given by

$$\begin{aligned}
&u_{tt} + a^2 u_{xxxx} = 0 \\
&u(x, 0) = f(x)\ ,\ u_t(x, 0) = 0 \\
&u(0, t) = u_x(0, t) = 0\ ,\ u(L, t) = u_{xx}(L, t) = 0
\end{aligned}$$

Question 1: By performing a separation of variables, show that the problem can be reduced to the following ODEs:

$$\begin{aligned}
X_{xxxx} - \lambda X &= 0 \\
T_{tt} + a^2 \lambda T &= 0
\end{aligned}$$

and determine the associated boundary conditions for $X(x)$.

Question 2: Show that $\lambda \geq 0$. (Hint: consider multiplying the spatial ODE by X, and integrating over the interval $[0, L]$).

Question 3: Show that the solution for X is

$$X(x) = A\cosh\mu x + B\sinh\mu x + C\cos\mu x + D\sin\mu x$$

where $\mu^4 = \lambda$.

Question 4: Show that the boundary conditions imply that

$$\begin{aligned}
C &= -A \\
D &= -B \\
\tan\mu L &= \tanh\mu L \\
B &= -\frac{A}{\tanh\mu L}
\end{aligned}$$

Question 5: Using a suitable diagram, show that there exist an infinity of solutions μ_n satisfying the above equation with $\mu_n > 0$, and determine an approximation to μ_n for large n.

Question 6: Solve the temporal ODE and deduce that

$$u(x, t) = \sum_{n=1}^{\infty} \alpha_n \cos(a\mu_n^2 t) X_n(x)$$

Question 7: Prove that the eigenfunctions $X_n(x)$ are orthogonal to each other with respect to the following inner product:

$$< u, v >= \int_0^L u(x)v(x)\mathrm{d}x$$

(Hint: use a method similar to that used to prove orthogonality in a regular Sturm Liouville problem – do not attempt to prove it directly!).

Question 8: Deduce an integral expression for the coefficients α_n in terms of $f(x)$.

Problem 6: Sturm-Liouville problem for an elliptic equation

Consider the problem

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial z^2} = 0 \text{ where } r \in [0, a] \ , \ z > 0$$
$$u(r, 0) = f(r) \ , \ |u(r, x)| < +\infty$$

Question 1: Describe in words or with a suitable diagram what physical problem this could correspond to.

Question 2: Using a separation of variables, show that the problem can be reduced to the following ODEs:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = -\lambda R$$
$$\frac{\partial^2 Z}{\partial z^2} = \lambda Z$$

and give a suitable physical or mathematical argument showing that $\lambda > 0$.

Question 3: Show that the spatial ODE can be cast in the form of a Bessel equation

$$x^2\frac{\partial^2 R}{\partial x^2} + x\frac{\partial R}{\partial x} + x^2 R = 0$$

for a suitable change of variable between r and x.

Question 4: The above Bessel equation has two solutions, $J_0(x)$ and $Y_0(x)$, shown in Figure 5. Given all of the above boundary and regularity conditions, deduce that the solution to the problem is given by

$$u(r, z) = \sum_{n=0}^{\infty} \alpha_n J_0(\sqrt{\lambda_n} r)e^{-\sqrt{\lambda_n} z}$$

What transcendental equation do the eigenvalues λ_n satisfy?

Question 5: Write the expression for the relevant inner product in the space of spatial eigenfunctions, and show that the eigenfunctions are orthogonal. (Hint: use a method similar to that used to prove orthogonality in a regular Sturm Liouville problem – do not attempt to prove it directly!).

Question 6: Given the formula

$$\int_0^1 J_0^2(\mu x)x\mathrm{d}x = \frac{1}{2}J_1^2(\mu)$$

valid for any μ solution of the equation $J_0(\mu) = 0$, show that

$$\alpha_n = \frac{2}{a^2}\frac{1}{J_1^2(a\sqrt{\lambda_n})}\int_0^a rf(r)J_0(\sqrt{\lambda_n} r)\mathrm{d}r$$