3.3.1 Canonical form of Hyperbolic equations

Consider a hyperbolic eq. $\mathcal{L}(u) = a u_{xx} + 2b u_{xy} + c u_{yy} + d u_x + e u_y + f = g$
To change it into its canonical form we require a coordinate transform $(x,y) \to (\xi, \eta)$ such that

$$A = C = 0 \qquad \text{(in the notation of 3.2)}$$

$$\Leftrightarrow \begin{cases} a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0 \\ a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0 \end{cases}$$

→ two equations are equivalent

Now rewrite $a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = a\left[\xi_x^2 + \frac{2b}{a}\xi_x\xi_y + \frac{c}{a}\xi_y^2\right]$

provided $a \neq 0$ $= a\left[\left(\xi_x + \frac{b}{a}\xi_y\right)^2 + \frac{c}{a}\xi_y^2 - \frac{b^2}{a^2}\xi_y^2\right]$

$$= a\left[\left(\xi_x + \frac{b}{a}\xi_y\right)^2 - \frac{b^2}{a^2}\xi_y^2\left(1 - \frac{c}{a}\frac{a^2}{b^2}\right)\right]$$

$$= a\left[\left(\xi_x + \frac{b}{a}\xi_y\left(1+\sqrt{1-\frac{ca}{b^2}}\right)\right) \cdot \left(\xi_x + \frac{b}{a}\xi_y\left(1-\sqrt{1-\frac{ca}{b^2}}\right)\right)\right]$$

So $a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0$ if and only if

$$\begin{cases} \xi_x + \frac{b}{a}\left(1+\sqrt{1-\frac{ca}{b^2}}\right)\xi_y = 0 \\ \text{OR} \\ \xi_x + \frac{b}{a}\left(1-\sqrt{1-\frac{ca}{b^2}}\right)\xi_y = 0. \end{cases}$$

→ Let's choose ξ a solution of $\xi_x + \frac{b}{a}\left(1+\sqrt{1-ca/b^2}\right)\xi_y = 0$

η ——— $\eta_x + \frac{b}{a}\left(1-\sqrt{1-ca/b^2}\right)\eta_y = 0$

- ξ is constant on the characteristics defined from

$$\frac{dx}{d\tau} = 1 \qquad \frac{dy}{d\tau} = \frac{b}{a}\left(1+\sqrt{1-ca/b^2}\right)$$

or $dy/dx = \frac{b}{a}\left(1+\sqrt{1-ca/b^2}\right) = \frac{b+\sqrt{b^2-ac}}{a}$

and η is constant on characteristics satisfying

$$\frac{dy}{dx} = \frac{b}{a}\left(1 - \sqrt{1 - ac/b^2}\right)$$
$$= \frac{b - \sqrt{b^2 - ac}}{a}$$

3.3.2 Examples

① The wave equation

$$u_{tt} - c^2 u_{xx} = 0 \qquad \delta(\mathcal{L}) = c^2 > 0$$
$\rightarrow$ a hyperbolic equation

To find a coordinate system (ξ, η) in which the wave equation is reduced to its canonical form, we must solve

$$\xi_t^2 - c^2 \xi_x^2 = 0$$

$$\Leftrightarrow \quad (\xi_t - c\xi_x)(\xi_t + c\xi_x) = 0$$

let ξ be the solution of $\xi_t - c\xi_x = 0$
η ———————— $\eta_t + c\eta_x = 0$

ξ is constant on characteristics satisfying $\frac{dx}{dt} = -c$

$\Leftrightarrow \quad x = -ct + \text{constant}$

let ξ be that constant, so that $\xi(x,t) = x + ct$

η is constant on characteristics satisfying $\frac{dx}{dt} = c$

$\Leftrightarrow \quad x = ct + \text{constant}$

let η be that constant, so that $\eta(x,t) = x - ct$

In the new coordinate system, we veryfy that

$$u_{tt} - c^2 u_{xx} = -u_{\xi\eta} \cdot 4c^2 = 0$$

Indeed $\quad u_t = cu_\xi - cu_\eta \qquad u_{tt} = c^2 u_{\xi\xi} + c^2 u_{\eta\eta} - 2c^2 u_{\eta\xi}$

$\quad u_x = u_\xi + u_\eta \qquad u_{xx} = u_{\xi\xi} + u_{\eta\eta} + 2u_{\eta\xi}$

$\Rightarrow$ So $\quad -2c^2 u_{\eta\xi} - 2c^2 u_{\eta\xi} = 0 \Rightarrow \boxed{u_{\eta\xi} = 0}$

We can now find the solutions straightforwardly:

$$u_{\xi\eta} = 0 \iff u = F(\xi) + G(\eta)$$
$$= F(x+ct) + G(x-ct)$$

where F and G are chosen to satisfy the required boundary conditions.

SKIP

(2) The Tricomi equation

$$u_{xx} + x u_{yy} = 0$$

$\delta(\mathcal{L}) = -x$
So the equation is hyperbolic for $x<0$.

→ We restrict the following work to the $x<0$ domain.

We seek the change of variable $(x,y) \to (\xi,\eta)$ which will simplify it into a canonical form
⇒ we require
$$\xi_x^2 + x\xi_y^2 = 0 = \xi_x^2 - |x|\xi_y^2$$
$$\iff \left(\xi_x + \sqrt{|x|}\,\xi_y\right)\left(\xi_x - \sqrt{|x|}\,\xi_y\right) = 0$$

let ξ be the solution of $\xi_x + \sqrt{|x|}\,\xi_y = 0$

→ ξ is constant on characteristics determined from
$$\frac{dy}{dx} = \sqrt{|x|}$$
$$\iff \quad y = \frac{2}{3}|x|^{\frac{3}{2}} + \text{constant}$$

$$\text{So} \quad \xi = y - \frac{2}{3}|x|^{\frac{3}{2}}$$

Similarly for η, $\frac{dy}{dx} = -\sqrt{|x|}$ so
$$y = -\frac{2}{3}|x|^{\frac{3}{2}} + \text{constant}$$
$$\Rightarrow \eta = y + \frac{2}{3}|x|^{\frac{3}{2}}$$

then $\xi_y = 1$, $\xi_x = -|x|^{\frac{1}{2}}$ $\xi_{yy} = 0$ $\xi_{xx} = -\frac{1}{2|x|^{1/2}}$

$\eta_y = 1$ $\eta_x = |x|^{1/2}$ $\eta_{yy} = 0$ $\eta_{xx} = \frac{1}{2|x|^{1/2}}$.

so $u_x = \xi_x u_\xi + \eta_x u_\eta$ $u_y = u_\xi + u_\eta$

$= -|x|^{1/2} u_\xi + |x|^{1/2} u_\eta$

$$u_{xx} = -\frac{1}{2}|x|^{-\frac{1}{2}} u_\xi + \frac{1}{2}|x|^{-\frac{1}{2}} u_\eta - |x|^{\frac{1}{2}}\left[\xi_x u_{\xi\xi} + \eta_x u_{\xi\eta}\right]$$
$$+ |x|^{1/2}\left[\xi_x u_{\xi\eta} + \eta_x u_{\eta\eta}\right]$$
$$= |x|(u_{\xi\xi} + u_{\eta\eta}) - |x| u_{\xi\eta} + \frac{1}{2}|x|^{-\frac{1}{2}}(u_\eta - u_\xi)$$

$$u_{yy} = \left[\xi_y u_{\xi\xi} + \eta_y u_{\xi\eta} + \xi_y u_{\eta\xi} + \eta_y u_{\eta\eta}\right]$$
$$= u_{\xi\xi} + u_{\eta\eta} + 2u_{\eta\xi}$$

so

$$|x|(u_{\xi\xi} + u_{\eta\eta}) - |x| u_{\xi\eta} + \frac{1}{2}|x|^{-\frac{1}{2}}(u_\eta - u_\xi) - |x|(u_{\xi\xi} + u_{\eta\eta} + 2u_{\eta\xi})$$
$$= -3|x| u_{\xi\eta} + \frac{1}{2}|x|^{-\frac{1}{2}}(u_\eta - u_\xi) = 0$$
$$\Leftrightarrow -3|x|^{\frac{3}{2}} u_{\xi\eta} + \frac{1}{2}(u_\eta - u_\xi) = 0$$

but $|x|^{3/2} = \frac{3}{4}(\eta - \xi)$

so finally $u_{\xi\eta} + \frac{4}{9}\frac{1}{\xi - \eta} \cdot \frac{1}{2}(u_\eta - u_\xi) = 0$

is the canonical form of the Tricomi equation

3.3.2 Parabolic equations

To transform a parabolic equation into its canonical form, we require a change of coordinate acting such that

$$B = C = 0 \quad \text{(in the notation of 3.2)}$$

However since by definition $AC - B^2 = 0$, it is sufficient to require that $C = 0$

$\Rightarrow$ we need $\quad a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 = 0$

But now recall that $ac - b^2 = 0$ so this is a perfect square so that it can be rewritten as

$$a\left(\eta_x + \sqrt{\frac{c}{a}}\,\eta_y\right)^2 = 0$$

alternatively $\quad \frac{1}{a}\left(a\eta_x + b\eta_y\right)^2 = 0.$

$\Rightarrow$ we can take η solution of the first order PDE

$$a\eta_x + b\eta_y = 0$$

$\Rightarrow$ η constant on the characteristics defined by $\frac{dy}{dx} = \frac{b}{a}$

[Note that this time ξ can be any function of x and y such that the Jacobian of (ξ, η) doesn't vanish

Example: $\quad x^2u_{xx} - 2xy\,u_{yx} + y^2u_{yy} + xu_x + yu_y = 0$

$$\delta(\mathcal{L}) = x^2y^2 - x^2y^2 = 0$$

The characteristics satisfy $\frac{dy}{dx} = \frac{xy}{x^2} = -\frac{y}{x}$

So $\quad \ln y = -\ln x + \text{const}$

or $\quad y = \frac{K}{x} \quad \Rightarrow$ take $\eta = xy$ and for simplicity, $\xi = x$

$$u_x = u_\xi + y u_\eta = U_\xi + \frac{\eta}{\xi} U_\eta \quad \text{since} \quad \xi_x = 1 \quad \xi_y = 0$$

$$u_y = x u_\eta = \xi U_\eta \qquad \eta_x = y. \quad \eta_y = x$$

$$u_{xx} = u_{\xi\xi} + y u_{\xi\eta} + y^2 u_{\eta\eta} = U_{\xi\xi} + \frac{\eta}{\xi} U_{\eta\xi} + \left(\frac{\eta}{\xi}\right)^2 U_{\eta\eta}$$

$$u_{xy} = xy\, u_{\eta\eta} + x u_{\eta\xi} + u_\eta = \eta U_{\eta\eta} + \xi U_{\eta\xi} + U_\eta$$

$$u_{yy} = x^2 u_{\eta\eta} = \xi^2 U_{\eta\eta}$$

So we now have

$$\xi^2 \left[u_{\xi\xi} + 2\frac{\eta}{\xi} u_{\eta\xi} + \frac{\eta^2}{\xi^2} u_{\eta\eta} \right]$$

$$-2\eta \left[\eta u_{\eta\eta} + \xi u_{\eta\xi} + u_\eta \right]$$

$$+ \frac{\eta^2}{\xi^2} \left[\xi^2 u_{\eta\eta} \right] + \xi \left(u_\xi + \frac{\eta}{\xi} u_\eta \right) + \frac{\eta}{\xi} \left(\xi u_\eta \right) = 0$$

$$\Leftrightarrow \quad \xi^2 u_{\xi\xi} + \xi u_\xi = 0 \quad \Leftrightarrow \quad \boxed{u_{\xi\xi} + \frac{1}{\xi} u_\xi = 0}$$

$\rightarrow$ the canonical form required.

This is now a simple ODE for $v = \frac{\partial u}{\partial \xi}$:

$$v_\xi + \frac{1}{\xi} v = 0$$

$$\Rightarrow \frac{dv}{v} = -\frac{1}{\xi} d\xi \rightarrow \ln v = -\ln \xi + f(\eta)$$

$$\rightarrow v = \frac{\hat{f}(\eta)}{\xi}$$

then $u_\xi = \dfrac{\hat{f}(\eta)}{\xi} \Rightarrow u(\xi, \eta) = \ln \xi \cdot \hat{f}(\eta) + g(\eta)$

$$\Rightarrow u(x, y) = f(xy) \cdot \ln x + g(xy).$$

3.3.3 Canonical form for Elliptic equations

Given a second order linear PDE which is elliptic, to reduce it to its canonical form we must find a coordinate change $(x,y) \to (\xi,\eta)$ such that

$$\begin{cases} A = C \\ B = 0 \end{cases}$$

So we need $$\begin{cases} a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 \\ a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y = 0 \end{cases}$$

let's construct the complex quantity $\phi = \xi + i\eta$ then this system is equivalent to

$$a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 = 0$$

Indeed

$$\begin{aligned} a\phi_x^2 + 2b\phi_x\phi_y + c\phi_y^2 &= a(\xi_x + i\eta_x)^2 + 2b(\xi_x + i\eta_x)(\xi_y + i\eta_y) \\ &\quad + c(\xi_y + i\eta_y)^2 \\ &= a\xi_x^2 - a\eta_x^2 + 2b(\xi_x\xi_y - 2b\eta_x\eta_y \\ &\quad + c\xi_y^2 - c\eta_y^2 + i\big[2a\xi_x\eta_x + \\ &\quad 2b(\xi_x\eta_y + \eta_x\xi_y) + 2c\,\xi_y\eta_y\big] \end{aligned}$$

So equating real & imaginary parts to 0 recovers the required system.

$\Rightarrow$ Characteristic equations imply

$$\frac{dy}{dx} = \frac{b \pm i\sqrt{ac-b^2}}{a} \qquad \text{since } ac - b^2 < 0$$

however, this time the characteristics "live" in a "complex plane".

The characteristic equations are complex conjugates so their solutions (say ϕ and ψ) will also be c.c.s.

Once the solution is found, we recover ξ and η by taking

$$\xi = Re(\phi)$$
$$\eta = Im(\phi).$$

(Note: we can arbitrarily choose ϕ or ψ → the only difference is in the sign of η).

<u>Example</u>: the Tricomi equation $u_{xx} + xu_{yy} = 0$ for $x > 0$

then we solve

$$\frac{dy}{dx} = \pm i\sqrt{x} \Rightarrow dy = \pm i\sqrt{x}\, dx$$

so the solution is

$$\frac{3}{2}y = \pm i x^{3/2} + \text{constant} \quad \to \text{choose constant} = \phi$$

so let $\phi = \frac{3}{2}y \pm ix^{3/2}$

so $\begin{cases} \xi = \frac{3}{2}y \\ \eta = x^{3/2} \end{cases}$

then $\begin{cases} \xi_x = 0 & \xi_y = \frac{3}{2} \\ \eta_x = \frac{3}{2}x^{1/2} & \eta_y = 0 \quad \eta_{xx} = \frac{3}{4}x^{-1/2} \end{cases}$

so $u_{xx} + xu_{yy} = \frac{9}{4}x\, u_{\eta\eta} + \frac{3}{4}x^{-1/2}u_\eta$

$$+ x\left(\frac{9}{4}u_{\xi\xi}\right)$$

$$= 0 \qquad x = \left(\frac{2}{3}\eta_x\right)^2$$

$$\Rightarrow u_{\eta\eta} + u_{\xi\xi} + \frac{1}{3}x^{-3/2}u_\eta = 0$$

$$\Rightarrow u_{\xi\xi} + u_{\eta\eta} + \frac{1}{3\eta}u_\eta = 0 \quad \to \text{Canonical form of the equation for } x > 0$$

SUMMARY

When trying to find the canonical form of

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} + \mathcal{L}^{(1)}(u) = g(x,y)$$

① Construct $\frac{dy}{dx} = \frac{-b \pm \sqrt{b^2-ac}}{a}$, and solve this ODE.

② • if $b^2 - ac > 0$ then we get 2 equations, yielding two solutions ξ and η.

• if $b^2 - ac = 0$ then we get 1 equation for η. Then choose any ξ such that the mapping $(x,y) \to (\xi, \eta)$ is indeed a change of coordinates

• if $b^2 - ac < 0$ then we get two complex conjugate solutions ϕ and ϕ^*. Then

$$\xi = Re(\phi)$$

$$\eta = Im(\phi)$$

③ Express the PDE in the new coordinate system.

Note: Be careful about $b(x,y)$ (the factor of 2)

$\Rightarrow$ If you are unsure, note that if the PDE is written as

$$\alpha(x,y)u_{xx} + \beta(x,y)u_{xy} + \gamma(x,y)u_{yy} + \mathcal{L}^{(1)}(u) = g(x,y)$$

then $\frac{dy}{dx} = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$

and this is entirely equivalent to the previous case (since $\beta = 2b$, $\alpha = a$, $\gamma = c$).