2.5 Weak solutions, shocks and entropy condition

2.5.1 Example of Burger's equation

$u_t + uu_x = 0$
$u(x,0) = f(x)$

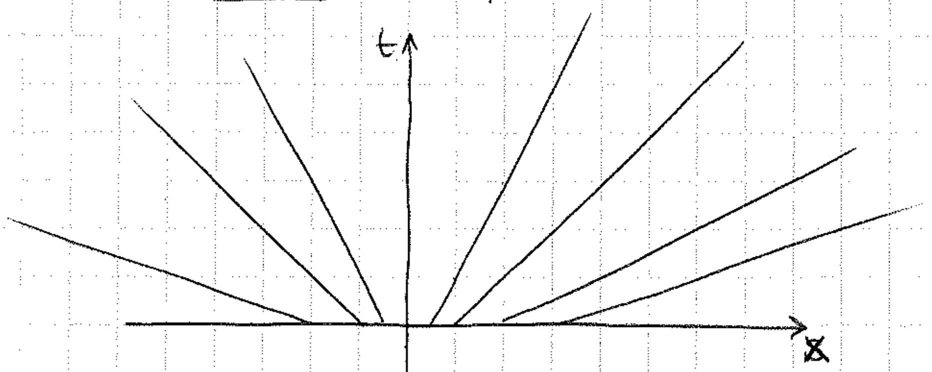
Characteristic equations: $\frac{dt}{d\tau} = 1 \rightarrow t = \tau$

$\frac{dx}{d\tau} = u \rightarrow x = u\tau + s$

$\frac{du}{d\tau} = 0 \rightarrow u = u_0(s) = f(s)$

Characteristics: Straight lines $t = \frac{x-s}{f(s)}$

Example 1: $f(s) = s$



Characteristics

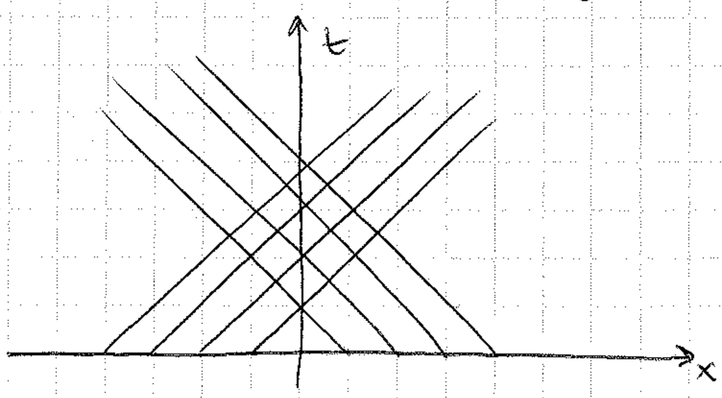
→ the solution exists at all times, no problem

$u = f(s) = s$
$= x - ut$

so $u = \frac{x}{1+t}$

Example 2: First type of problem: crossing characteristics

$$f(s) = -\frac{|s|}{s} = \begin{cases} 1 & \text{if } x < 0 \\ -1 & \text{if } x \geqslant 0 \end{cases}$$

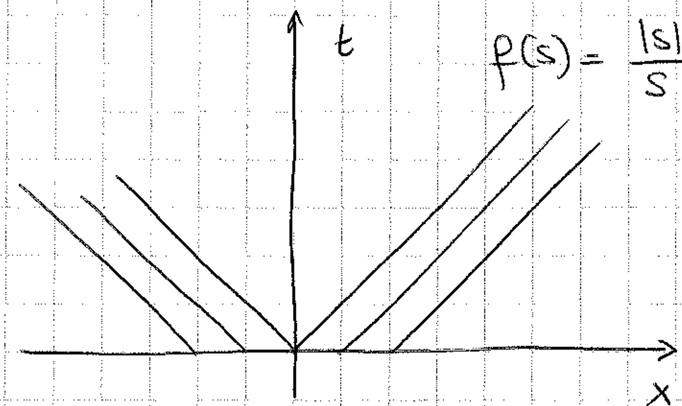


Characteristics intercect!

Since $u = u_0(s)$ is constant on characteristics, which value should we choose?!

Example 3 : Second type of problem: some region of space/time is not represented by any characteristic

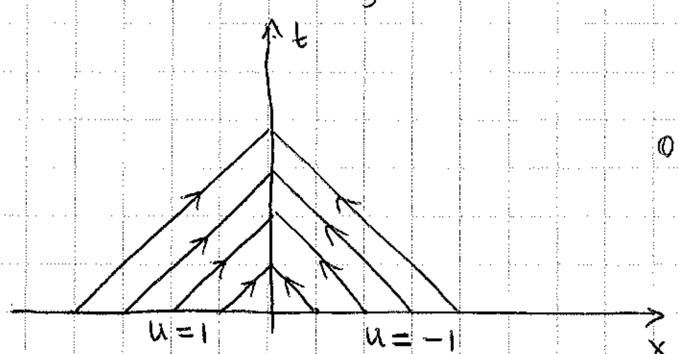
$$f(s) = \frac{|s|}{s} = \begin{cases} -1 \text{ if } x<0 \\ 1 \text{ if } x \geq 0 \end{cases}$$



What values should $u(x,t)$ take in the region which is not spanned by any characteristics?

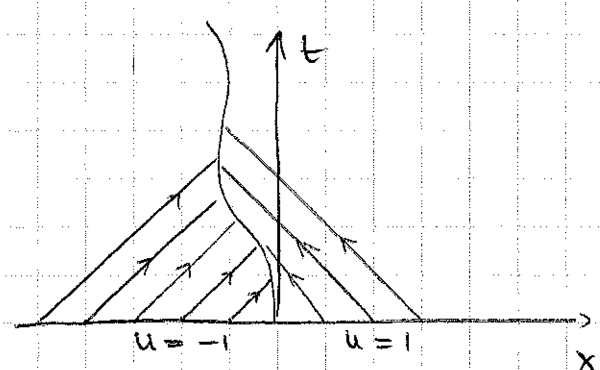
2.5.2 Weak problems and weak solutions

One way to resolve the first type of problem is to select a particular line separating characteristics emanating from the left & from the right and selecting the corresponding solution on each side



selecting x=0 line

or



selecting a random line...

Problems! • the solution then appears to be discontinuous at the line ⇒ SHOCK
• there is more than one solution

Non-smooth solutions are called weak solutions. Weak solutions are not solutions of the PDE since $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ are not defined at discontinuities.
Weak solutions are solutions of the associated weak problem

Definition: A weak problem is an integral reformulation of the PDE for which solutions can be discontinuous!

Note: there are many possible weak problems associated to a given PDE.

2.5.3 Weak problems and conservation laws

Conservation laws of the kind

$$\frac{\partial u}{\partial t} + \nabla \cdot F = 0$$

are usually derived in physical systems from integral relationships anyway →

$$\frac{\partial}{\partial t}\int_{\text{volume}} u\,dV + \int_{\text{surface}} F\cdot dS = 0 = \frac{\partial}{\partial t}\int_{\text{volume}} u\,dV + \int_{\text{volume}} \nabla\cdot F\,dV$$

so we may as well use these integral formulations as our weak problem.

Take $\quad u_t + \frac{\partial}{\partial x}\left[F(u)\right] = 0$

and integrate over an interval $[a,b]$ at a given time t:

$$\frac{\partial}{\partial t}\int_a^b u\,dx + \int_a^b \frac{\partial}{\partial x}\left[F(u)\right]dx = 0$$

$$\Leftrightarrow \quad \frac{\partial}{\partial t}\int_a^b u\,dx + F(u(b,t)) - F(u(a,t)) = 0 \qquad (*)$$

→ this is the weak formulation of a conservation law

- Any smooth solution of $(*)$ is also a solution of the associated PDE.
- However, we can now construct non-smooth solutions.

Assume the solution has one discontinuity in the solution $u(x,t)$ located on the line $x = \gamma(t)$ such that

$$\begin{cases} u(x,t) = u_-(x,t) & \text{if} \quad x < \gamma(t) \\ u(x,t) = u_+(x,t) & \text{if} \quad x > \gamma(t) \end{cases}$$

Then, plugging this into (*) we get

$$\frac{\partial}{\partial t}\left[\int_a^{\gamma(t)} u_-(x,t)\,dx + \int_{\gamma(t)}^{b} u_+(x,t)\,dx\right] + F(u(b,t)) - F(u(a,t)) = 0$$

Recall: $$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = \frac{db}{dt} f(b(t),t) - \frac{da}{dt} f(a(t),t) + \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}\,dx$$

So

$$\frac{\partial}{\partial t}\int_a^{\gamma(t)} u_-(x,t)\,dx = \frac{d\gamma}{dt} u_-(\gamma(t),t) + \int_a^{\gamma} \frac{\partial u_-}{\partial t}\,dx$$

$$\frac{\partial}{\partial t}\int_{\gamma(t)}^{b} u_+(x,t)\,dx = -\frac{d\gamma}{dt} u_+(\gamma(t),t) + \int_{\gamma(t)}^{b} \frac{\partial u_+}{\partial t}\,dx$$

So $$\frac{d\gamma}{dt}\left[u_-(\gamma(t),t) - u_+(\gamma(t),t)\right] + \int_a^{\gamma(t)} \frac{\partial u_-}{\partial t}\,dx + \int_{\gamma(t)}^{b} \frac{\partial u_+}{\partial t}\,dx$$
$$+ F(u(b,t)) - F(u(a,t)) = 0.$$

Now write

$$\begin{aligned} F(u(b,t)) - F(u(a,t)) &= F(u(b,t)) - F(u_+(\gamma(t),t)) + F(u_+(\gamma(t),t)) \\ &\quad + F(u_-(\gamma(t),t)) - F(u(a,t)) - F(u_-(\gamma(t),t)) \\ &= \int_a^{\gamma(t)} \frac{\partial}{\partial x}(F(u_-))\,dx + \int_{\gamma(t)}^{b} \frac{\partial}{\partial x}(F(u_+))\,dx + F(u_+(\gamma(t),t)) \\ &\quad - F(u_-(\gamma(t),t)) \end{aligned}$$

So finally we get

$$\int_a^{\gamma(t)} \cancel{\frac{\partial u_-}{\partial t} + \frac{\partial}{\partial x}\left(F(u_-)\right)}\, dx + \int_{\gamma(t)}^{b} \cancel{\frac{\partial u_+}{\partial t} + \frac{\partial}{\partial x}\left(F(u_+)\right)}\, dx$$

$$+ \frac{d\gamma}{dt}\Big(u_-(\gamma(t),t) - u_+(\gamma(t),t)\Big) + F(u_+(\gamma(t),t)) - F(u_-(\gamma(t),t)) = 0$$

$$\Rightarrow \quad \frac{d\gamma}{dt} = \frac{F(u_+(\gamma(t),t)) - F(u_-(\gamma(t),t))}{u_+(\gamma(t),t) - u_-(\gamma(t),t)}$$

An equation for the discontinuity curve (shock curve) in terms of the jump in u and $F(u)$ across the shock. Sometimes written as

$$\frac{d\gamma}{dt} = \frac{[F]}{[u]}$$

Rankine-Hugoniot jump condition.

To find $\gamma(t)$ we need an initial condition: take $\gamma(t_c) = x_c$ where t_c is the earliest time (with $t_c > 0$) for which characteristics cross and x_c is the position at which this happens.

<u>Example 1</u> Burgers equation $\left(F(u) = \frac{u^2}{2}\right)$ with $f(s) = -\frac{|s|}{s}$.

We saw the earliest (positive) characteristics crossing occurs at $x_c = 0$, $t_c = 0$

On the left side of $\gamma(t)$, $u = u_- = 1$
— right —, $u = u_+ = -1$

$F(u_+) = \frac{1}{2}$ $\quad F(u_-) = \frac{1}{2}$ so

$$\frac{d\gamma}{dt} = \frac{0}{2} = 0 \Rightarrow \gamma = \text{constant} \Rightarrow \gamma = 0$$

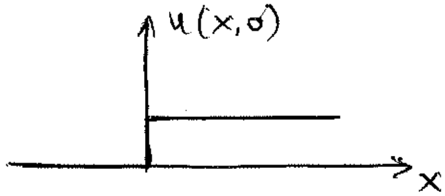
So the correct discontinuity line is $\boxed{x = 0}$.

Example 2 Traffic problems:

Suppose we try to solve for a traffic flow

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(F(u)\right) = 0 \quad \text{with} \quad F(u) = V_0 u\left(1 - \frac{u}{U_{max}}\right)$$

$$u(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{U_{max}}{4} & \text{if } x > 0 \end{cases}$$

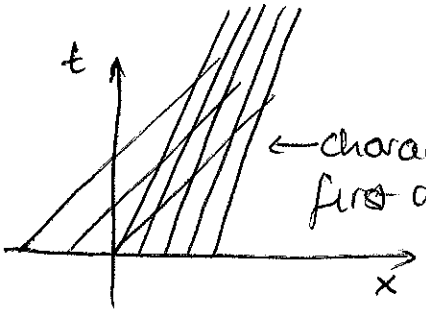


The characteristics are given by

$$x = F'(\phi(s))t + s \quad \text{where} \quad \phi(s) = 0 \text{ if } s < 0, \quad \phi(s) = \frac{U_{max}}{4} \text{ if } s > 0$$

so

$$\begin{cases} x = V_0 t + s & \text{if } s < 0 \\ x = \frac{V_0}{2}t + s & \text{if } s > 0 \end{cases}$$



←characteristics first cross at $x=0$, $t=0$

on left: $u_- = 0$

on right: $u_+ = \frac{U_{max}}{4}$

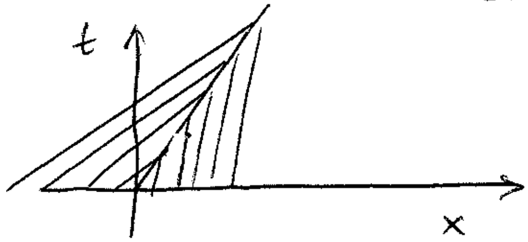
$F(u_-) = 0$

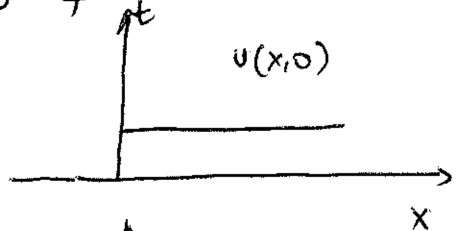
$F(u_+) = V_0 \cdot \frac{U_{max}}{4}\left(1 - \frac{1}{4}\right) = V_0 \frac{3U_{max}}{16}.$

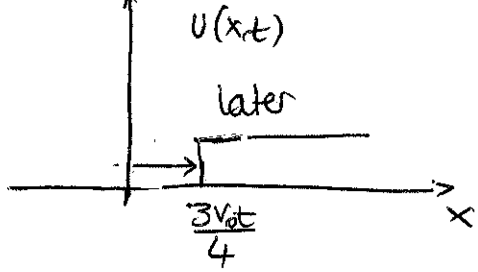
So $$\frac{d\gamma}{dt} = \frac{[F]}{[u]} = \frac{3V_0}{4}$$

$\Rightarrow$ since $\gamma(t=0) = 0$ then $\gamma(t) = \frac{3V_0 t}{4}$

So the shock travels at velocity $\frac{3V_0}{4}$







Homework

Repeat same problem with

$$u(x,0) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{3U_{max}}{4} & \text{if } x > 0. \end{cases}$$

2.5.4 Crossing of Characteristics, initial shock position (x_c, t_c)

For conservation laws $\begin{cases} u_t + [F(u)]_x = 0 \\ u(x,0) = \phi(x) \end{cases}$

the characteristics are straight lines with $t = \dfrac{x-s}{F'(\phi(s))}$

Two lines emanating from s_1 and s_2 on the initial condition curve have equations

$$\begin{cases} t = \dfrac{x-s_1}{F'(\phi(s_1))} & \Rightarrow \quad x = tF'(\phi(s_1)) + s_1 \\[2ex] t = \dfrac{x-s_2}{F'(\phi(s_2))} & \Rightarrow \quad x = tF'(\phi(s_2)) + s_2 \end{cases}$$

$\Rightarrow$ they intercept at time t_+ satisfying

$$t_+F'(\phi(s_1)) + s_1 = t_+F'(\phi(s_2)) + s_2$$

$$\Leftrightarrow \quad t_+ = \frac{s_2 - s_1}{F'(\phi(s_1)) - F(\phi(s_2))}$$

So t_c is the minimum value of t_+ over all possible values of s_1 and s_2 (such that $t_+ \geq 0$)

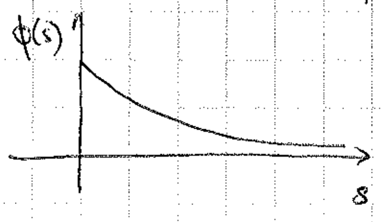
$$t_c = \min_{s_1, s_2} \frac{s_2 - s_1}{F'(\phi(s_1)) - F'(\phi(s_2))} \quad \text{with } t_c \geq 0.$$

<u>Note</u>: IF $F'(\phi(s_1)) \leq F'(\phi(s_2))$ while $s_2 \geq s_1$ then t_c cannot be ≥ 0

$\Rightarrow$ if $\dfrac{d}{ds}[F'(\phi(s))] > 0$ for all s then characteristics never intercept for $t \geq 0$.

<u>Example 1</u> Traffic flow: $F'(\phi(s)) = V_0\left(1 - \dfrac{2\phi(s)}{V_{max}}\right)$

- if $\phi(s) = \dfrac{V_{max}}{4} e^{-s}$



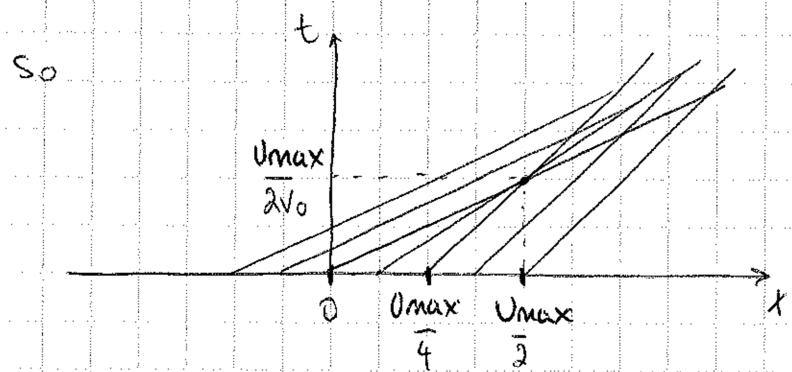
$$F'(\phi(s)) = V_0\left(1 - \frac{e^{-s}}{2}\right)$$

so $\dfrac{d}{ds}[F'(\phi(s))] = \dfrac{V_0}{2} e^{-s} \geq 0 \Rightarrow$ no shock.

Example 2

$$\phi(s) = \begin{cases} 0 & \text{if } s \leq 0 \\ s & \text{if } 0 \leq s \leq \frac{U_{max}}{4} \\ \frac{U_{max}}{4} & \text{if } s \geq \frac{U_{max}}{4} \end{cases}$$

$$\rightarrow \quad F'(\phi(s)) = \begin{cases} V_0 & \text{if } s \leq 0 \\ V_0\left(1 - \frac{2s}{U_{max}}\right) & \text{if } 0 \leq s \leq \frac{U_{max}}{4} \\ V_0/2 & \text{if } s \geq \frac{U_{max}}{4} \end{cases}$$



Can we confirm this mathematically?

Select $s_2 > s_1$

- if $s_2 > s_1$, $s_1 \leq 0$ and $s_2 \leq 0 \Rightarrow$ no crossing
- if $s_2 > s_1$, $s_1 \leq 0$, $s_2 \in [0, \frac{U_{max}}{4}]$ then

$$t_+ = \frac{s_2 - s_1}{V_0 - V_0\left(1 - \frac{2s_2}{U_{max}}\right)} = \frac{s_2 - s_1}{\frac{2V_0 s_2}{U_{max}}} = \frac{U_{max}}{2V_0}\,\frac{s_2 - s_1}{s_2}$$

this is minimized when $s_1 \rightarrow 0$

- if $s_2 > s_1$, $s_1 \in [0, \frac{U_{max}}{4}]$, $s_2 \geq \frac{U_{max}}{4}$ then

$$t_+ = \frac{s_2 - s_1}{V_0\left(1 - \frac{2s_1}{U_{max}}\right) - \frac{V_0}{2}} = \frac{s_2 - s_1}{\frac{V_0}{2} - \frac{2V_0 s_1}{U_{max}}}$$

this is minimized when $s_2 \rightarrow \frac{U_{max}}{4}$

$$\rightarrow \quad t_c = \min_{s, s_2} t_+ = \frac{U_{max}}{2V_0} \qquad x_c = F'(\phi(0))\,t_c = \frac{U_{max}}{2}$$ as seen on diagram.

2.5.5 Traffic flow revisited

From the example above, we now try to solve for $\gamma(t)$:

$$\frac{d\gamma}{dt} = \frac{F(u_+) - F(u_-)}{u_+ - u_-} \qquad \text{where } F(u) = v_0 u\left(1 - \frac{u}{U_{max}}\right)$$

- Characteristics from the left have $u_- = 0$
- ———————— right ——— $u_+ = U_{max}/4$

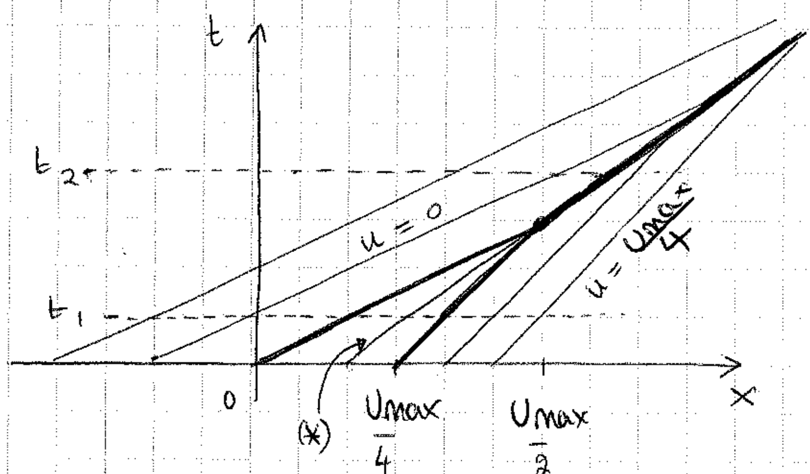
So
$$\frac{d\gamma}{dt} = \frac{v_0 \frac{U_{max}}{4}\left(1 - \frac{1}{4}\right) - 0}{\frac{U_{max}}{4} - 0} = \frac{3v_0}{4}\ldots$$

So
$$\gamma(t) - x_c = \frac{3v_0}{4}(t - t_c)$$

$$\Rightarrow \gamma(t) = \frac{U_{max}}{2} + \frac{3v_0}{4}\left(t - \frac{U_{max}}{2v_0}\right)$$

$$= \frac{3v_0}{4}t + \frac{U_{max}}{8}$$

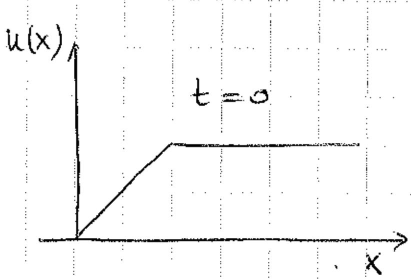
So now we can construct the solution (graphically)

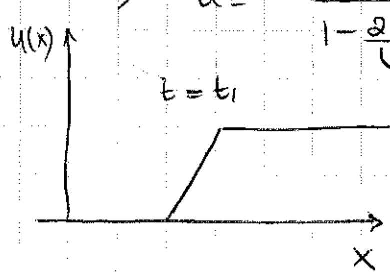


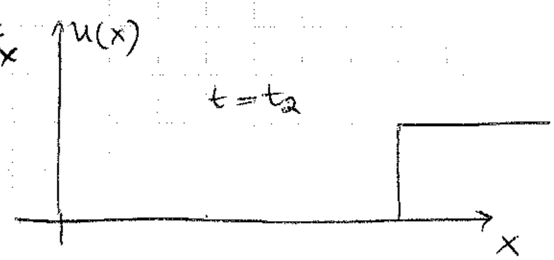
in (*):
$$u = \phi(x - F'(\phi(\,))t) = x - v_0\left(1 - \frac{2u}{U_{max}}\right)t$$

So
$$u\left(1 - \frac{2v_0}{U_{max}}t\right) = x - v_0 t$$

$$\Rightarrow u = \frac{x - v_0 t}{1 - \frac{2v_0 t}{U_{max}}}$$







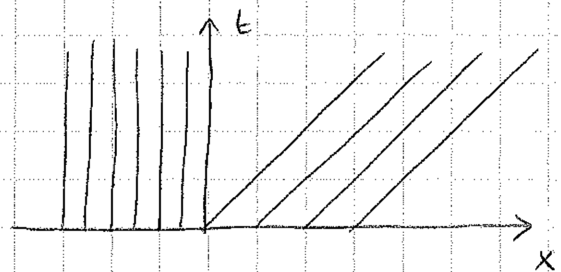
2.6.6 Expansion shocks and entropy condition

Consider the example

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = \Theta(x) \end{cases} \qquad (F(u) = \tfrac{1}{2}u^2)$$

$\leftarrow$ Heaviside function: $\begin{cases} \Theta(x) = 0 & x \le 0 \\ \Theta(x) = 1 & x \ge 0 \end{cases}$

Characteristics



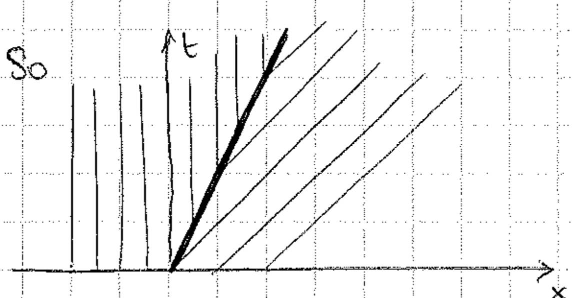
$$\begin{cases} t = \tau \\ x = u_0(s)\tau + s \\ u = u_0(s) = \Theta(s) \end{cases}$$

We could consider constructing a weak solution $u = u_- = 0$ on the left of $\gamma(t)$ and $u = u_+ = 1$ on the right of $\gamma(t)$.

The R.H. condition implies $\dfrac{d\gamma}{dt} = \dfrac{F(u_+) - F(u_-)}{u_+ - u_-} = \dfrac{1}{2}$

$\Rightarrow$ here $\gamma(t) = \frac{1}{2}t$

So



$$\begin{cases} u = 0 & x \le t/2 \\ u = 1 & x \ge t/2 \end{cases}$$

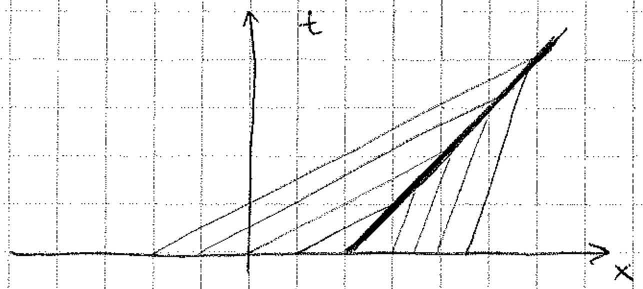
Problem: this solution is not physically acceptable because it is not causal: information appears to be "created" on the discontinuity and is then carried by the characteristics.

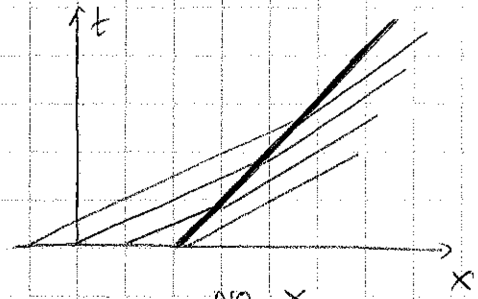
In other words: we would like the system to be entirely determined by its initial conditions, not by arbitrary extensions of the solution

Definition The entropy condition

Characteristics must enter the discontinuity (the shock front) but are not allowed to emanate from it

To guarantee this, the slope of the characteristics on the left must be shallower than $\gamma(t)$, and those on the right steeper:





$$\Rightarrow \quad \frac{1}{F'(u_-)} \leq \frac{1}{\frac{d\gamma}{dt}} \leq \frac{1}{F'(u_+)}$$

$$\Leftrightarrow \quad \boxed{F'(u_-) \geq \frac{d\gamma}{dt} \geq F'(u_+)}$$

Problem: How do we construct solutions if $F'(u)$ is an increasing function of u? (see example above).

Going back to the characteristic solutions:

$$\begin{cases} t = \tau \\ x = F'(\phi(s))t + s \\ u = \phi(s) \end{cases} \quad \text{to} \quad \begin{cases} u_t + [F(u)]_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

Assume characteristics diverge from $s = s_0$. At this point,

$$x = F'(\phi(s_0))t = F'(u)t + s_0$$

so let's construct

$$u = G\left(\frac{x - s_0}{t}\right)$$

where G is the inverse function of F'.

and use this as a solution in the "fan" region

Example 1 $\quad u_t + u u_x = 0 \qquad F(u) = \frac{1}{2}u^2 \quad F'(u) = u$

with $\quad u(x,0) = \Theta(x)$

Characteristics diverge from $s=0$, so that

$$x = ut \quad \text{or} \quad u = x/t \quad \text{in the "fan"}$$

$\Rightarrow$ we construct the weak solution with

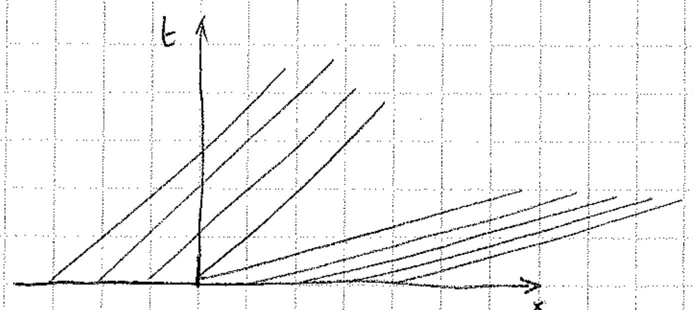
$$\begin{cases} u = 0 & \text{if } x \leq 0 \\ u = x/t & \text{if } 0 \leq x \leq t \\ u = 1 & \text{if } x \geq t \end{cases}$$

Example 2: $\quad u_t + (e^u)_x = 0 \quad$ with $\quad u(x,0) = \Theta(x)$

then $\quad u_t + e^u u_x = 0$

$\rightarrow$ characteristics are

$$t = \frac{x-s}{e^{\Theta(s)}} = \begin{cases} \frac{x-s}{1} & \text{if } s \leq 0 \\ \frac{x-s}{e} & \text{if } s \geq 0 \end{cases}$$



So let's construct from $x = e^u t + s$ the solution

$u = \ln(x/t)$ emanating from $s=0$

$$\Rightarrow \begin{cases} u = 0 & \text{if } x \leq t \\ u = \ln(x/t) & \text{if } t \leq x \leq et \\ u = 1 & \text{if } x \geq et \end{cases}$$

Check: in region $t \leq x \leq et$

$$\frac{\partial u}{\partial t} = -\frac{x}{t^2} \cdot \frac{t}{x} = -\frac{1}{t} \qquad e^u \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(e^u) = \frac{\partial}{\partial x}\left(\frac{x}{t}\right) = \frac{1}{t}$$

$$\Rightarrow \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}(e^u) = 0 \text{ as required}$$