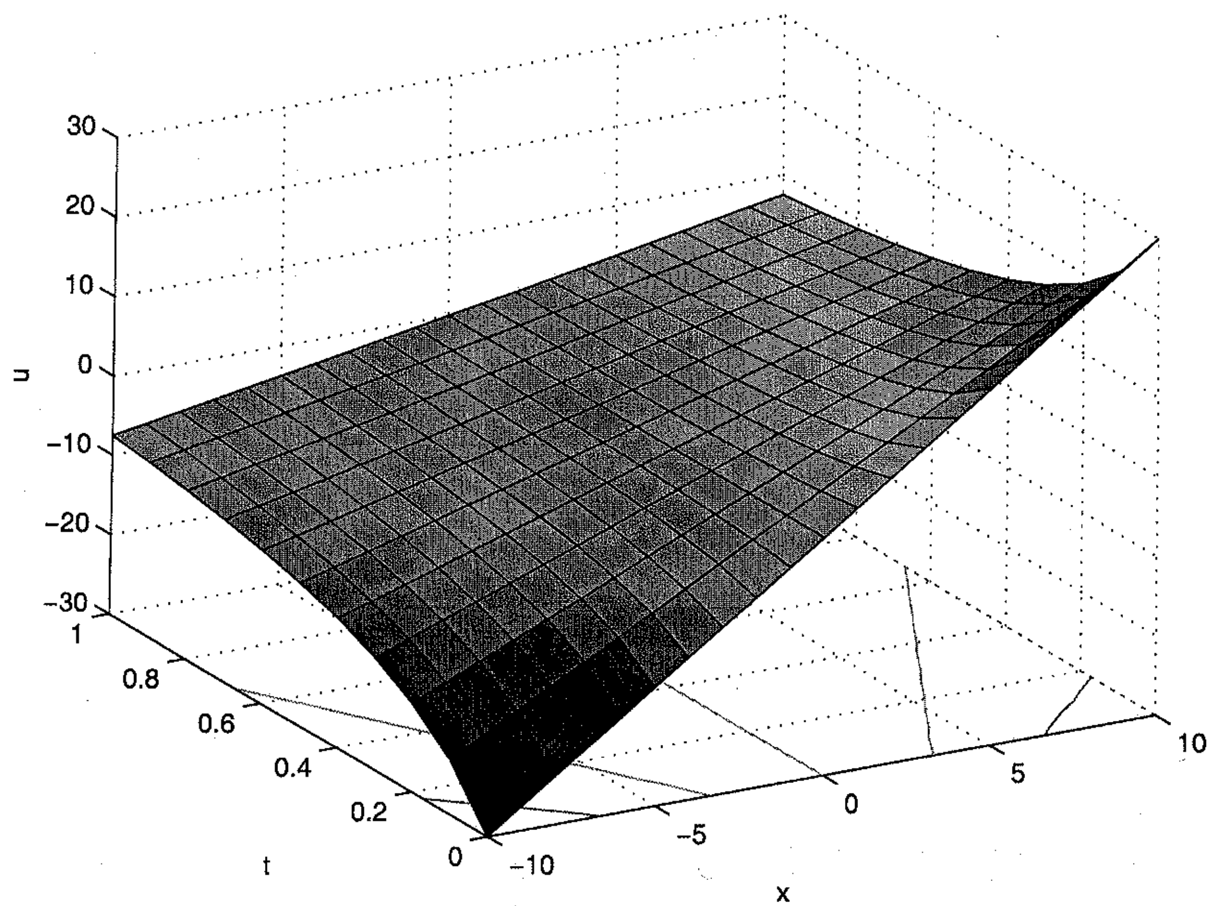
Solution to

$u_t + u u_x = 0$

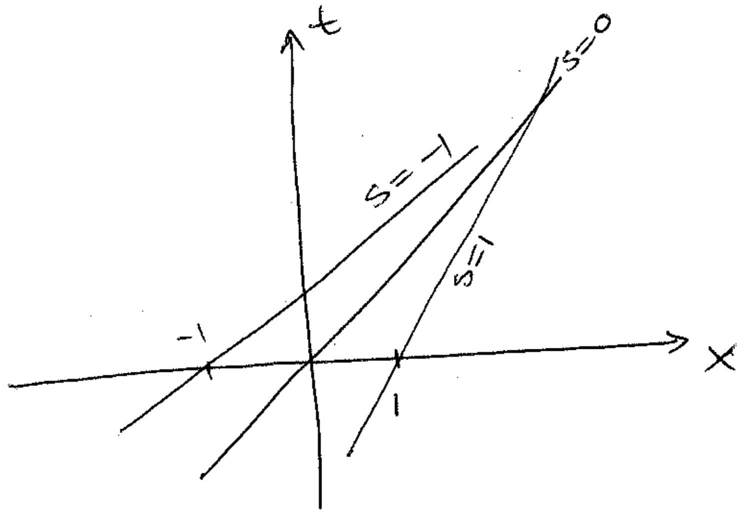
$u(x, 0) = 3x$



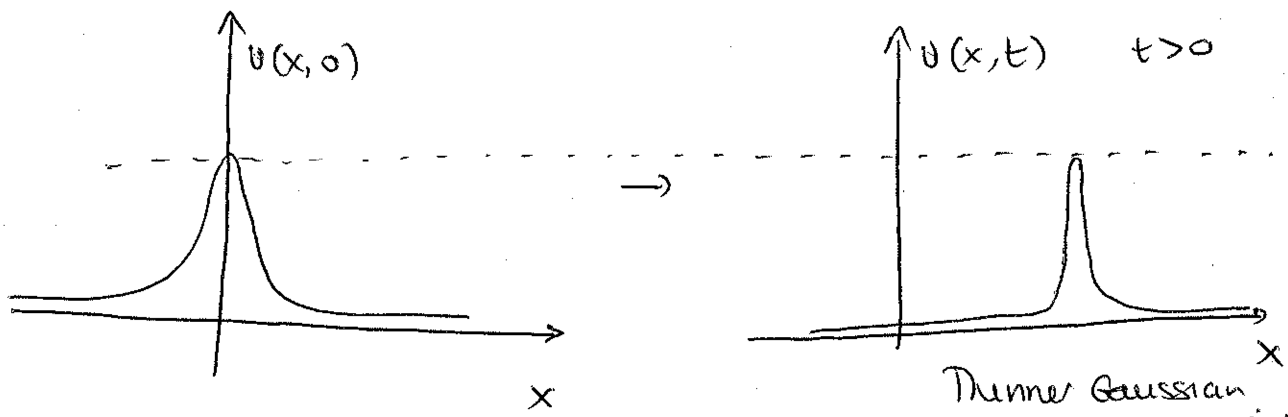
However, it is possible still to find out approximately what the look like based on the characteristics:

$$x = F'(\phi(s))t + s \Rightarrow t = \frac{x-s}{F'(\phi(s))}$$

Here $F'(u) = u$
$\phi(s) = e^{-s^2/2}$ $\Rightarrow$ $t = \dfrac{x-s}{e^{-s^2/2}} = (x-s)e^{s^2/2}$

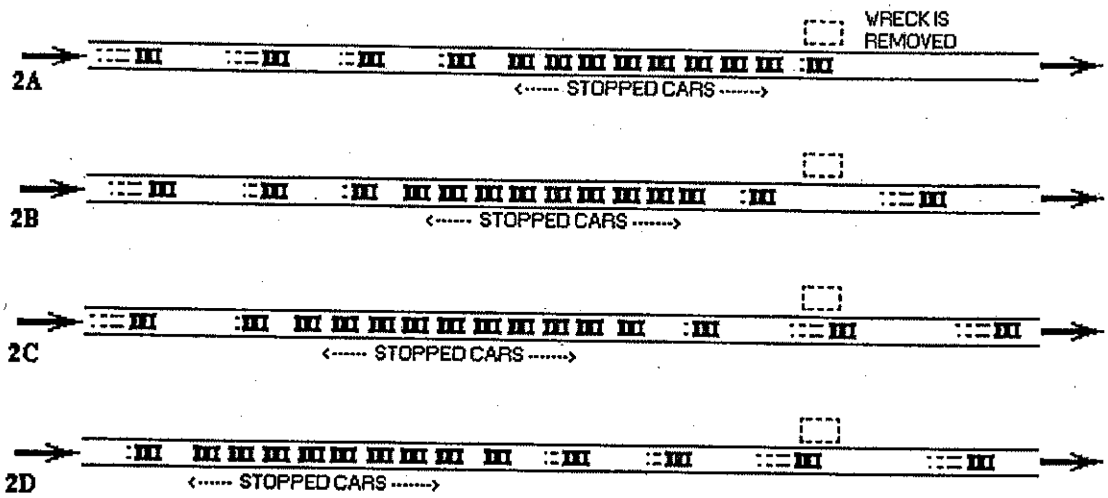


$\Rightarrow$ We see that the solution is transported to the right and "squeezed":



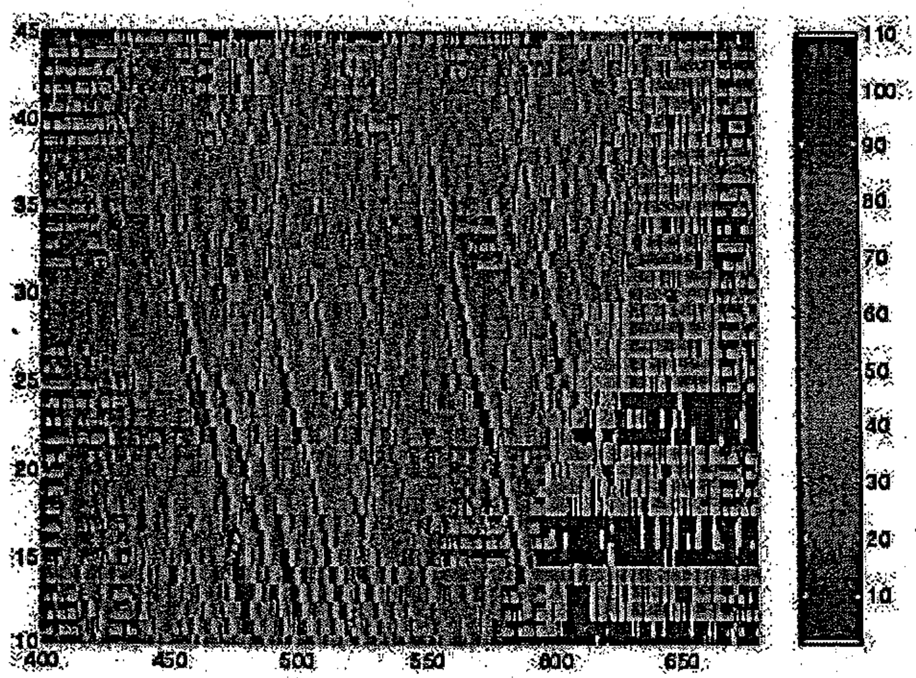
Thinner Gaussian but same amplitude

Illustration of the cause of traffic waves



M25 data (London) of traffic density vs space & time

time

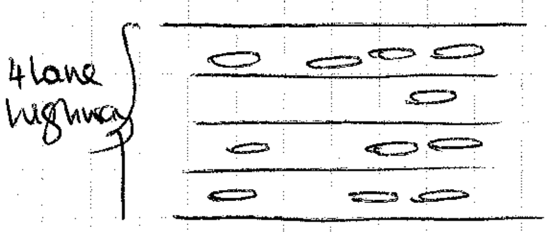


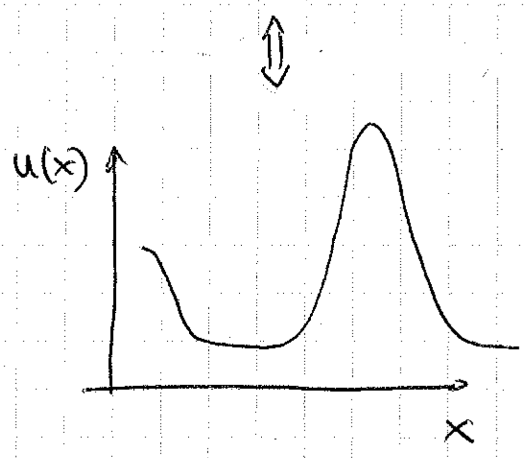
distance

2.4.4 Traffic flow

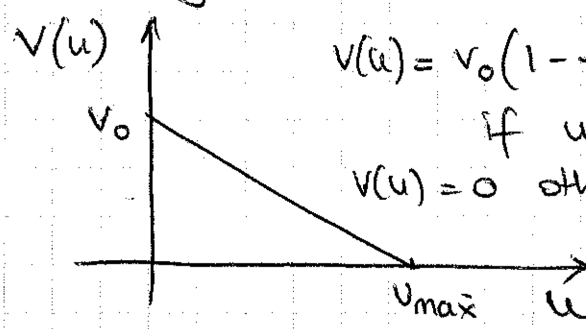
The study of traffic flow is an attempt at modelling (for example) the flow of cars on a road / highway, but also for example of information on a network, etc..

Idea: ① Model the road/network as a 1D line, with some density $u(x,t)$ of traffic (cars/information packets) at time t, position x.





② Model the velocity of the traffic flow as a function of the traffic density:



$$V(u) = V_0\left(1 - \frac{u}{U_{max}}\right) \quad \text{if } u \in [0, U_{max}]$$

$$V(u) = 0 \quad \text{otherwise}$$

→ Flowing traffic has optimal velocity V_0 when u is small, and stalls when $u > U_{max}$

The conservation law for the car density $u(x,t)$ is simply

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(u V(u)\right) = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[V_0 u\left(1 - \frac{u}{U_{max}}\right)\right] = 0$$

So here we have a conservation law with

$$F(u) = V_0 u \left(1 - \frac{u}{U_{max}}\right) \quad \text{if } 0 < u < U_{max}$$

$$= 0 \quad \text{otherwise}$$

$$\Rightarrow \quad F'(u) = V_0\left(1-\frac{u}{U_{max}}\right) - \frac{V_0 u}{U_{max}}$$

$$F'(u) \begin{cases} = V_0\left(1-\frac{2u}{U_{max}}\right) & \text{if} \quad u \in [0, U_{max}] \\ = 0 & \text{otherwise} \end{cases}$$

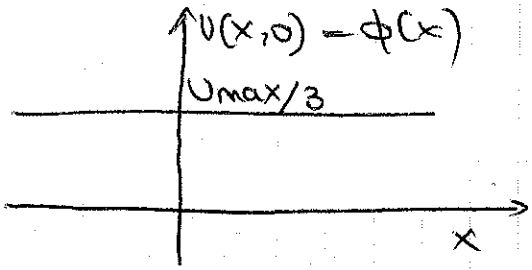
- The solution to any initial traffic condition $u(x,0) = \phi(x)$ is given by the algebraic equation

$$u(x,t) = \phi\left(x - F'(u)t\right).$$

- The solution $u(x,t)$ is constant along characteristics, which are straight lines with slope

$$\frac{1}{F'(\phi(s))}$$

<u>Example 1</u> Suppose we start with a uniform density of cars $u(x,0) = \frac{U_{max}}{3}$ $\forall x$.



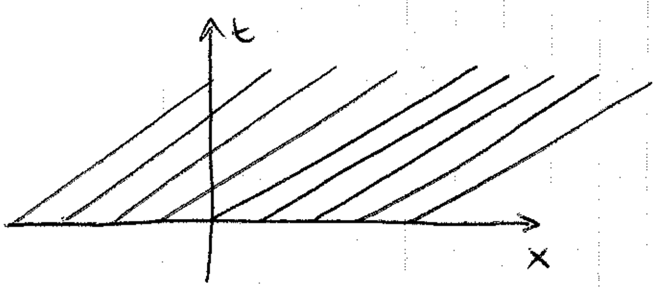
so $\phi(s) = \frac{U_{max}}{3}$ $\forall s$.

The characteristics are straight lines with equation

$$x = F'(\phi(s))t + s$$

$$\Leftrightarrow \quad x = F'\left(\frac{U_{max}}{3}\right)t + s$$

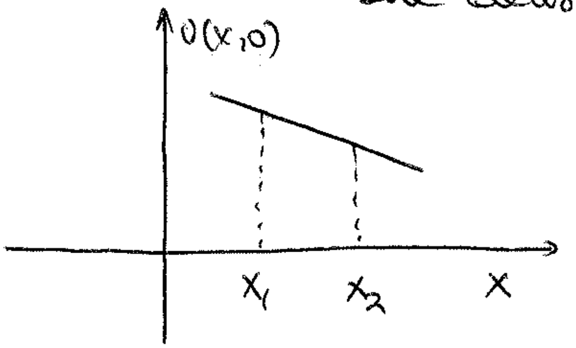
$$\Leftrightarrow \quad x = \frac{V_0}{3}t + s$$

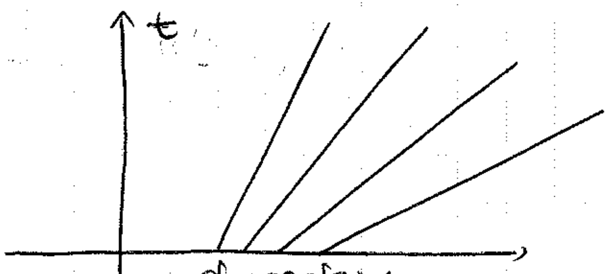


$\rightarrow$ since u is constant along a characteristic, we see that traffic is smoothly flowing at velocity $\frac{V_0}{3}$ and

$u(x,t) = \frac{U_{max}}{3}$ is always constant

Example 2 Suppose there is a local decrease in the density of cars with x $\left(v(x,0) < \frac{v_{max}}{2} \right)$





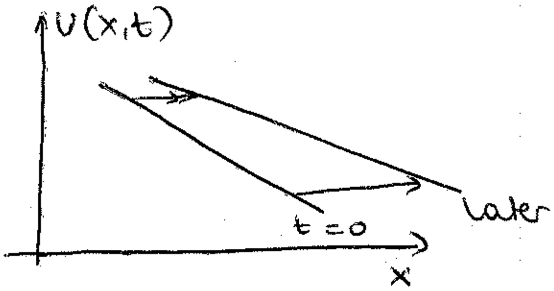
characteristics

$$v(x_2,0) < v(x_1,0)$$

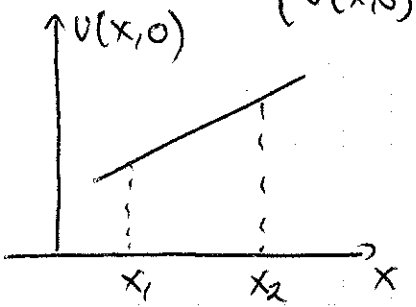
then $\phi(s_2) < \phi(s_1)$

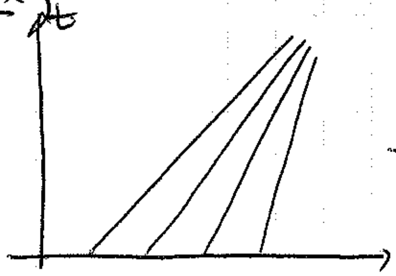
$\Rightarrow F'(s_2) > F'(s_1)$ → the slope of the characteristics in the (x,t) plane decreases

$\Rightarrow \dfrac{1}{F'(s_2)} < \dfrac{1}{F'(s_1)}$ regions of less dense traffic move faster

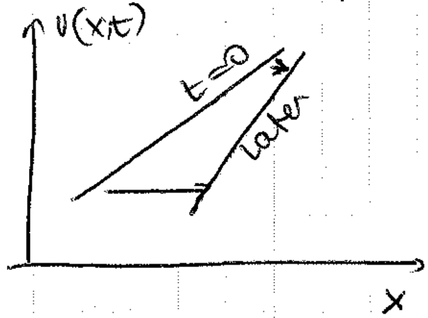


Example 3 Suppose there is a local increase in traffic $\left(v(x,0) < \frac{v_{max}}{2} \right)$





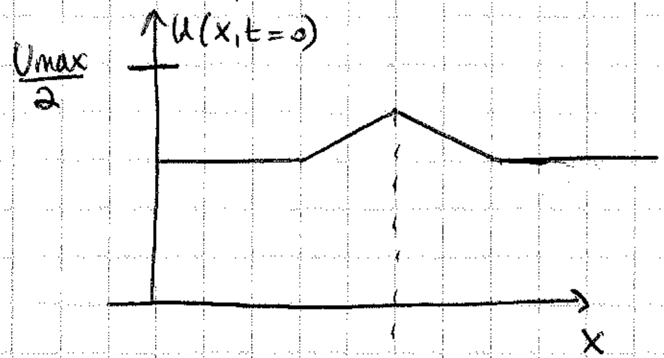
characteristics

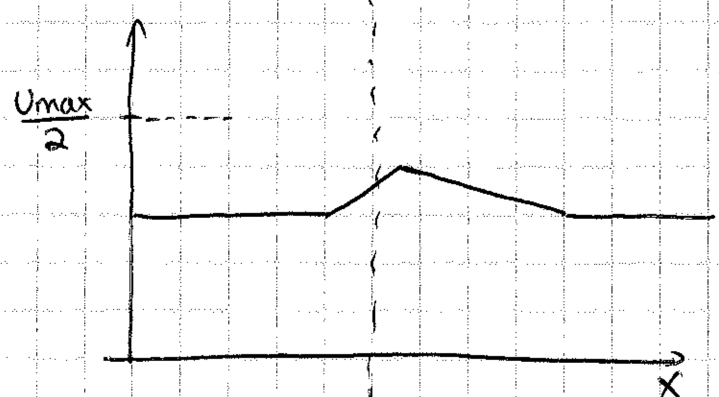


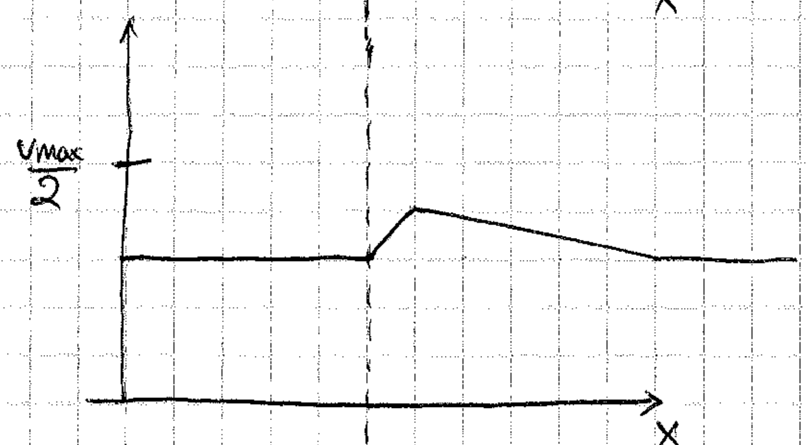
$\Rightarrow$ This behavior leads to the emergence of <u>traffic waves</u> spontaneously.

The traffic wave moves forward or backward depending on the density of traffic compared with $\frac{v_{max}}{2}$.

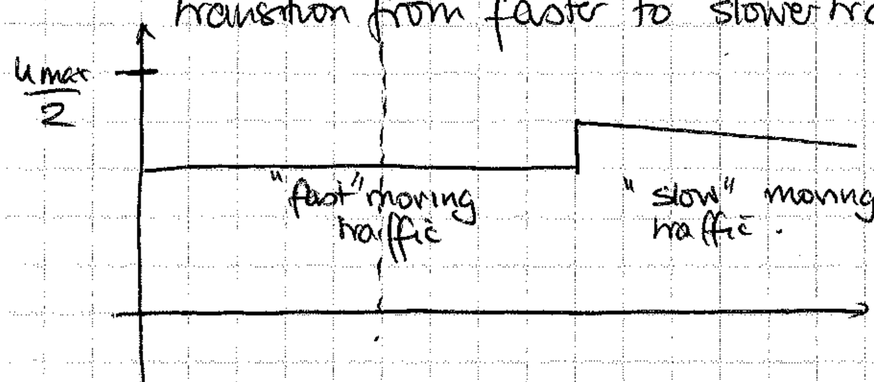
So if there is initially a small perturbation in the traffic density (but with $u(x,t=0) < \frac{u_{max}}{2}$ for all x)







until eventually a discontinuity forms with a transition from faster to slower traffic



<u>HOMEWORK</u> What happens if $u(x,t=0)$ exceeds $\frac{u_{max}}{2}$ locally?