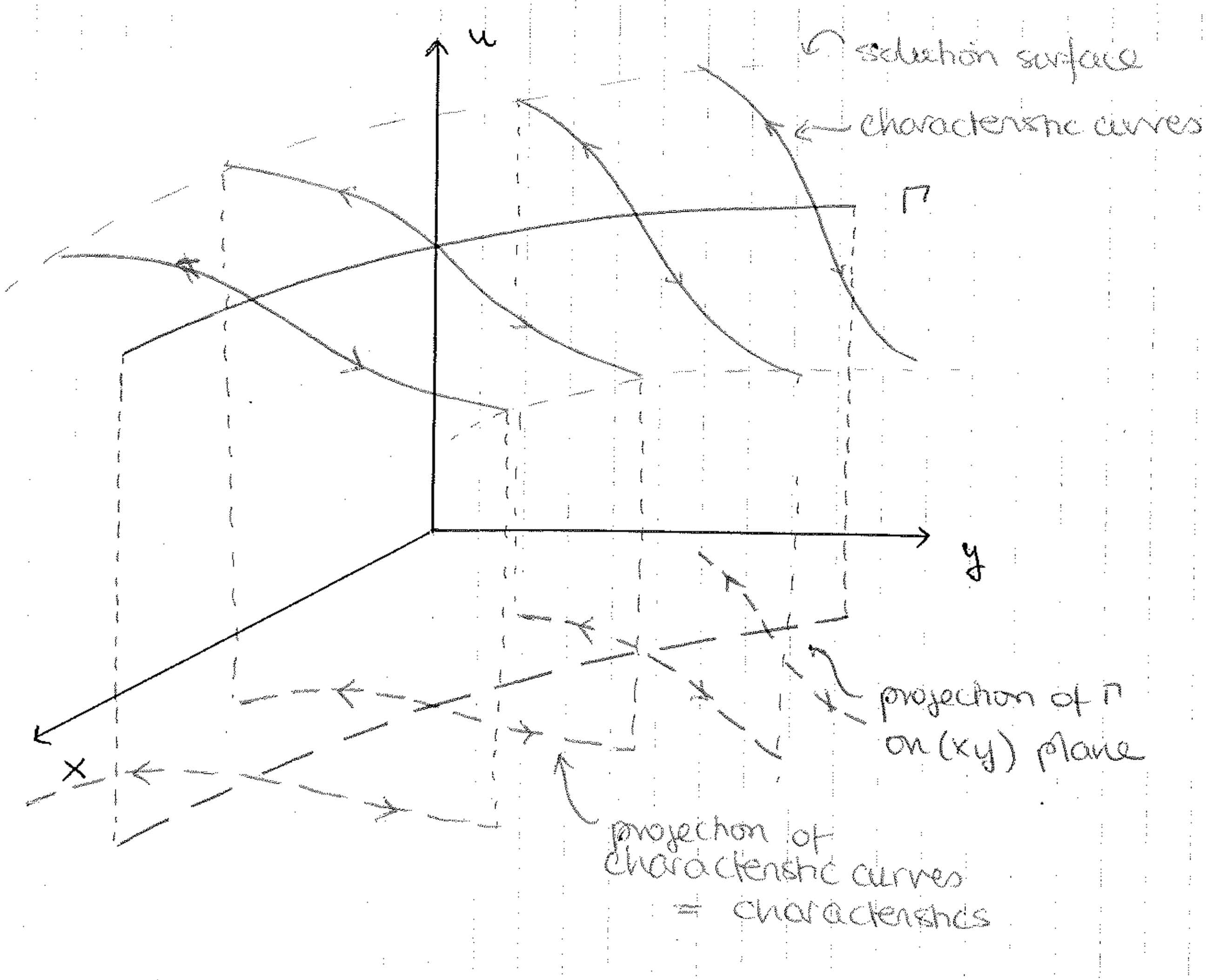
2.3.2 Existence and uniqueness theorem

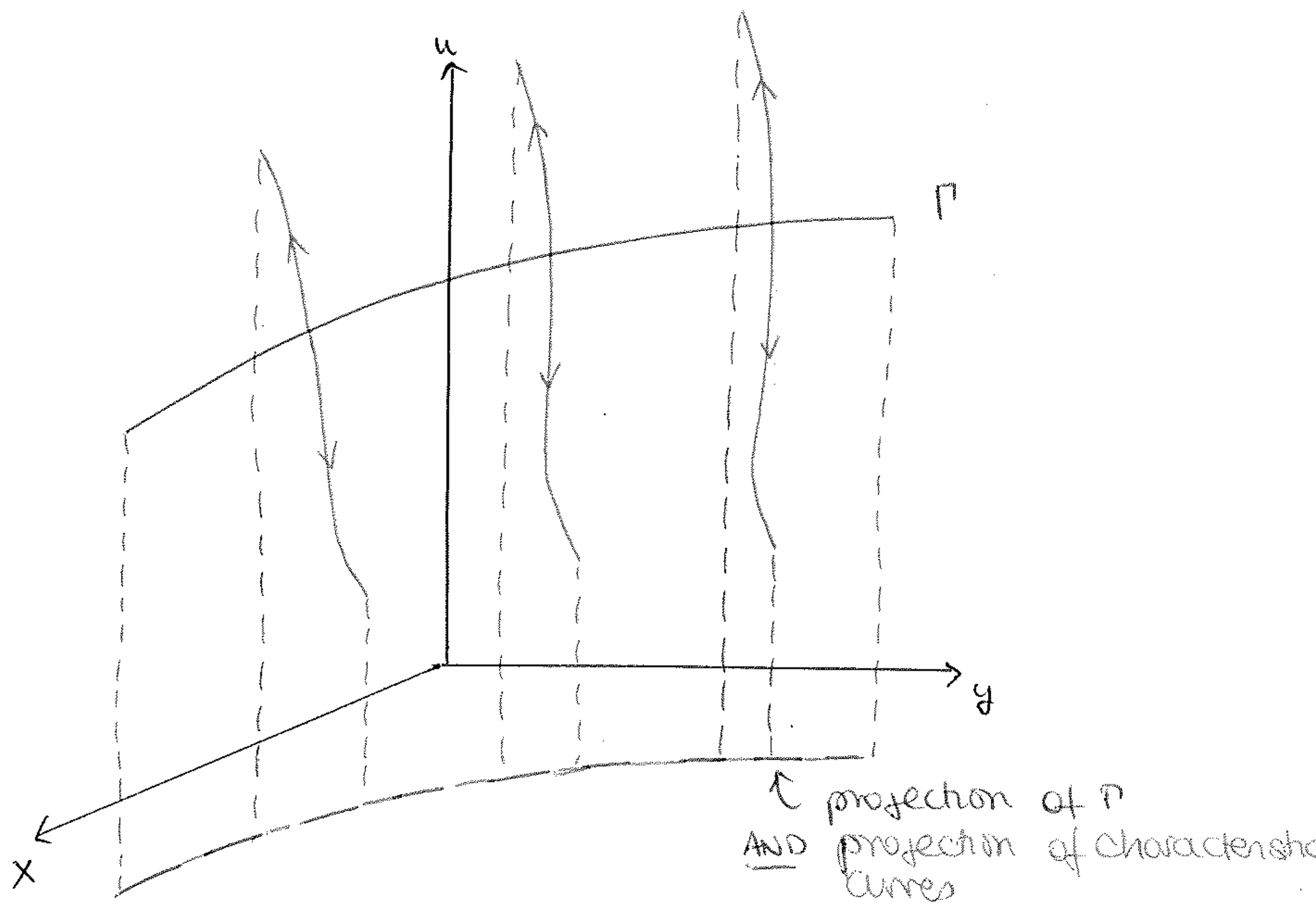
- In a given well-posed problem (solution exists and is unique), consider the surface defined as the set of points $(x, y, u(x,y))$ in the (x,y,u) space
- This surface contains Γ (initial condition curve)
- This surface is spanned by the characteristic curves. The solution $u(x,y)$ is propagated along the characteristic curves away from Γ.



- The projection of Γ and of the characteristic curves on the $(x-y)$ plane shows the characteristics intersecting the projection of Γ
  - → solution is indeed propagated away from Γ

• In an ill-posed problem, two situations may arise

Case 1: no solutions to the PDE



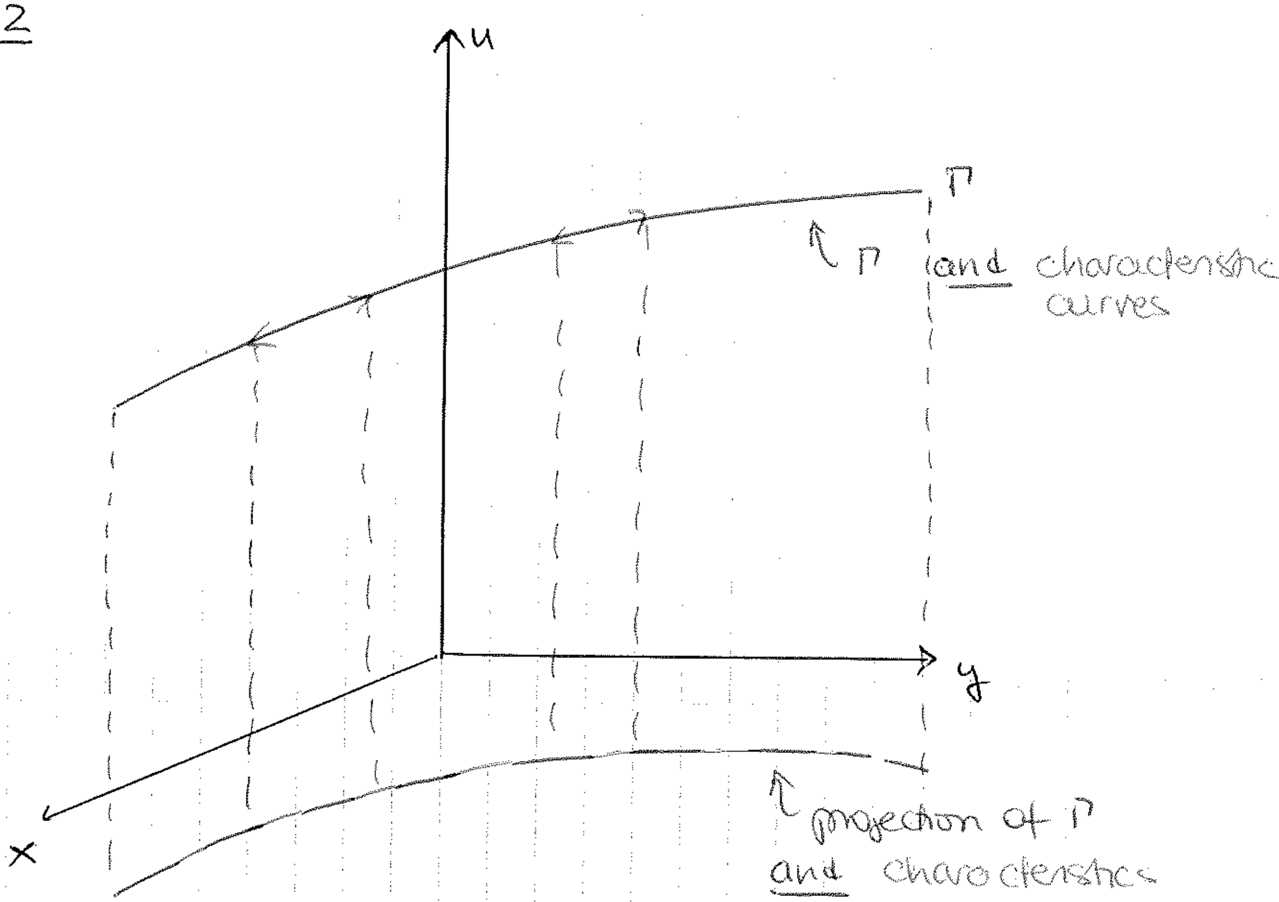
In this case: characteristic curves intercept the Γ curve, but the surface spanned by Γ and the characteristic curves entirely projects on a single curve in the (x,y) plane

⇒ the solution is not propagated away from Γ in a well-defined way

→ for every pt (x,y) on the projection of Γ ∃ an infinite # of values of u coming from each characteristic curve

⇒ here there are no solutions to the PDE

Case 2



⇒ This time the only constraint on the solution is that the surface $(x, y, u(x,y))$ must pass through Γ. Although the characteristic curves do not propagate the solution away from Γ, they do lie on Γ.

⇒ any surface which passes through Γ is a solution of the PDE

⇒ there are an ∞ # of solutions to the PDE.

⇒ The difference between the well-posed case and the ill-posed cases is clearly seen in the projection of Γ and the projection of the characteristic curves (the characteristics)

- If the characteristics intercept the projection of Γ ⇒ well posed problem

• If the characteristics are // to the projection of P

$\Rightarrow$ ill posed problem.

Mathematically

Two vectors in the x-y plane intersect (i.e are not //) provided they have non-zero cross product.

At a point s on the initial curve, the tangent vector is
$$\begin{pmatrix} dx_0/ds \\ dy_0/ds \\ du_0/ds \end{pmatrix}$$

$\Rightarrow$ its projection on (x-y) is $\begin{pmatrix} dx_0/ds \\ dy_0/ds \\ 0 \end{pmatrix}$

The characteristic curve emanating from s has tangent vector
$$\begin{pmatrix} dx/d\tau \\ dy/d\tau \\ du/d\tau \end{pmatrix}_{(x_0,y_0,u_0)} = \begin{pmatrix} a(x_0,y_0,u_0) \\ b(x_0,y_0,u_0) \\ c(x_0,y_0,u_0) \end{pmatrix} \rightarrow \text{its projection is } \begin{pmatrix} a(x_0,y_0,u_0) \\ b(x_0,y_0,u_0) \\ 0 \end{pmatrix}$$

The transversality condition @ a point s is therefore satisfied provided
$$\begin{pmatrix} dx_0/ds \\ dy_0/ds \\ 0 \end{pmatrix} \times \begin{pmatrix} a(x_0,y_0,u_0) \\ b(x_0,y_0,u_0) \\ 0 \end{pmatrix} \neq 0$$

$$\Leftrightarrow \quad b(x_0,y_0,u_0)\frac{dx_0}{ds} - a(x_0,y_0,u_0)\frac{dy_0}{ds} \neq 0$$

<u>Theorem</u>: • Assume that $a(x,y,u)$, $b(x,y,u)$ and $c(x,y,u)$ are smooth functions in a neighborhood of the initial curve (x_0, y_0, u_0)

• Assume that the transversality condition holds for each $s \in [s_0 - 2\delta, s_0 + 2\delta]$ on the initial curve

then: $\exists$ a unique solution $u(x,y)$ in the neighborhood of the initial curve defined by $\tau \in [-\epsilon, \epsilon]$, $s \in [s_0 - \delta, s_0 + \delta]$

Idea behind the proof

- given a system of ODEs for the characteristic curves

$$\begin{cases} \frac{\partial x}{\partial z} = a(x,y,u) \\ \frac{\partial y}{\partial z} = b(x,y,u) \\ \frac{\partial u}{\partial z} = c(x,y,u) \end{cases}$$

we can always find a solution that satisfies the initial conditions $\begin{cases} x(z=0) = x_0(s_0) \\ y(z=0) = y_0(s_0) \\ u(z=0) = u_0(s_0) \end{cases}$ from a point s_0 on the initial curve

in a neighborhood of $z=0$ (properties of dynamical systems) provided a, b & c are smooth functions near (x_0, y_0, u_0).

$\Rightarrow$ We can always find $\begin{cases} x(z,s) \\ y(z,s) \\ u(z,s) \end{cases}$ in a neighborhood of $z=0$, $s=s_0$

provided the initial condition curve is continuous near s_0.

- The problem of existence and uniqueness lies in the inversion of the system to obtain $u(x,y)$

Let's write
$$x(z,s) = x(0,s_0) + z\left(\frac{\partial x}{\partial z}\right)_{\substack{z=0 \\ s=s_0}} + (s-s_0)\left(\frac{\partial x}{\partial s}\right)_{\substack{z=0 \\ s=s_0}}$$

$$y(z,s) = y(0,s_0) + z\left(\frac{\partial y}{\partial z}\right)_{\substack{z=0 \\ s=s_0}} + (s-s_0)\left(\frac{\partial y}{\partial s}\right)_{\substack{z=0 \\ s=s_0}}$$

This is also

$$x = \underbrace{x_0(s_0)}_{\text{from initial conditions}} + z\,\underbrace{a(x_0,y_0,u_0)}_{\text{from PDE \& characteristic equation}} + (s-s_0)\underbrace{\left(\frac{\partial x_0}{\partial s}\right)_{s=s_0}}_{\text{from initial condition}}$$

and
$$y = y_0(s_0) + z\,b(x_0,y_0,u_0) + (s-s_0)\left(\frac{\partial y_0}{\partial s}\right)_{s=s_0}$$

Now to invert these equations to obtain τ and s in terms of x and y we have the matrix equation

$$\begin{pmatrix} a(x_0,y_0,u_0) & \frac{\partial x_0}{\partial s}\Big|_{s_0} \\ b(x_0,y_0,u_0) & \frac{\partial y_0}{\partial s}\Big|_{s_0} \end{pmatrix} \begin{pmatrix} \tau \\ s \end{pmatrix} = \begin{pmatrix} x - x_0(s_0) + s_0 \frac{\partial x_0}{\partial s} \\ y - y_0(s_0) + s_0 \frac{\partial y_0}{\partial s} \end{pmatrix}$$

$\Rightarrow$ this system has a unique solution provided

$$\begin{vmatrix} a(x_0,y_0,u_0) & \frac{\partial x_0}{\partial s} \\ b(x_0,y_0,u_0) & \frac{\partial y_0}{\partial s} \end{vmatrix} \neq 0 .$$

<u>As required</u>

<u>Example</u> Given the PDE

$$x u u_x + y u u_y = u^2 - 1$$

with the initial condition $u(x,x^2) = x^3$ for $x \in [a,b]$

for what values of (a,b) will there be a unique solution?

- initial condition curve $x_0(s) = s$
  $y_0(s) = s^2$
  $u_0(s) = s^3$

$$a(x_0,y_0,u_0) = x_0 u_0 = s^4$$
$$b(x_0,y_0,u_0) = y_0 u_0 = s^5$$
$$\frac{\partial x_0}{\partial s} = 1 \qquad \frac{\partial y_0}{\partial s} = 2s$$

$$\Rightarrow \begin{vmatrix} a & \frac{\partial x_0}{\partial s} \\ b & \frac{\partial y_0}{\partial s} \end{vmatrix} = \begin{vmatrix} s^4 & 1 \\ s^5 & 2s \end{vmatrix} = 2s^5 - s^5 = s^5$$

$\Rightarrow$ as long as $s \neq 0$ then $\exists$ a unique solution. So any interval <u>excluding</u> $s=0$ will lead to a unique solution

Exercise: find the solution for $(a,b) = (0,+\infty)$
(be careful with absolute values!)

2.4 Examples of use of quasilinear 1st order PDEs

2.4.1 Conservation laws (general)

- Conservation laws are usually written in the form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(F(u, x, t)\right) = 0$$

(or, in more than one dimension

$$\frac{\partial u}{\partial t} + \nabla \cdot \left(F(u, \underline{r}, t)\right) = 0 \; .)$$

- Why is this called a conservation law?

$\Rightarrow$ Expressed in their integral form, the conservation laws describes the conservation of the quantity u:

$$\int_D \frac{\partial u}{\partial t}\, d^3\underline{r} + \int_D \nabla \cdot \left(F(u, \underline{r}, t)\right) d^3\underline{r} = 0$$

$$\Leftrightarrow \quad \underbrace{\frac{\partial}{\partial t}\int_D u \, d^3\underline{r}}_{\text{total change of the quantity } u \text{ in the domain } D} + \underbrace{\int_{S = \text{contour of } D} F(u, \underline{r}, t)\, d^2\underline{r}}_{\text{Flux through the surface of the domain}} = 0$$

Example: The standard equation for the conservation of mass in a flow stirred by a velocity field $v(\underline{r})$ is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho\, v(\underline{r})\right) = 0$$

It can be rewritten as

$$\int_D \frac{\partial \rho}{\partial t} d\underline{r}^3 + \int_D \nabla \cdot (\rho v(r))\, d^3\underline{r}$$

$$\Leftrightarrow \quad \frac{\partial m}{\partial t} + \int_{\substack{\text{surface}\\ \text{of } D}} \rho v(\underline{r})\, d^2\underline{r} = 0$$

change of mass within volume D

mass carried by velocity field across surface of D

- Conservation laws occur in most domains in science.

2.4.2 Conservation law (specific)

Here for simplicity we consider conservation laws which can be written as

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(F(u)\right) = 0 \\ u(x,0) = \phi(x) \end{cases}$$

(the flux is a function of u only)

Then we see that

$$\frac{\partial u}{\partial t} + F'(u)\frac{\partial u}{\partial x} = 0$$

$\rightarrow$ a quasilinear homogeneous PDE that can be integrated with

$$\begin{cases} \frac{\partial t}{\partial \tau} = 1 \quad \longrightarrow \quad t = \tau \\ \frac{\partial x}{\partial \tau} = F'(u) \qquad (*) \\ \frac{\partial u}{\partial \tau} = 0 \quad \rightarrow \quad u \text{ is constant along characteristics: } u = u_0(s) = \phi(s) \end{cases}$$

So from (*) we see that

$$\frac{\partial x}{\partial \tau} = F'(\phi(s)) \implies x = F'(\phi(s))\,\tau + x_0(s)$$

in other words

$$\left\{\begin{array}{l} \tau = t \\ x = F'(\phi(s))\cdot t + s \\ u = \phi(s) \end{array}\right\} \text{ so } \quad u = \phi\left(x - F'(u)t\right)$$

<u>Interpretation</u>

① u is constant on characteristics

② The characteristics are <u>straight lines</u> in the $(x-t)$ plane with slope $\dfrac{1}{F'(\phi(s))}$, which depends only on the initial condition (and on F'.)

③ The problem of finding $u(x,t)$ becomes equivalent to solving the algebraic equation

$$u = \phi\left(x - F'(u)t\right)$$

It is sometimes possible to invert this analytically.

<u>2.4.3 Burger's inviscid equation (Euler's equation)</u>

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

$$\Leftrightarrow \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \quad \rightarrow \text{ a conservation law with } F(u) = \frac{u^2}{2}$$

$$F'(u) = u$$

so the solution for any given initial condition $u(x,0) = \phi(x)$ can formally be written as

$$u(x,t) = \phi\left(x - u(x,t)\,t\right)$$

which may or may not be solvable analytically

(1) Suppose

$$\phi(x) = 3x \quad \text{then}$$

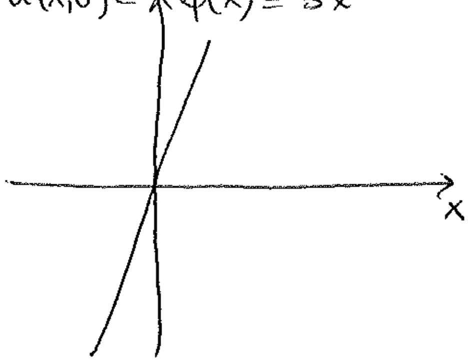
$$u(x,t) = 3\left(x - u(x,t)t\right)$$

$$\Leftrightarrow \quad u(x,t)\left[1 + 3t\right] = 3x$$

$$\Rightarrow \quad u(x,t) = \frac{3x}{1+3t}$$

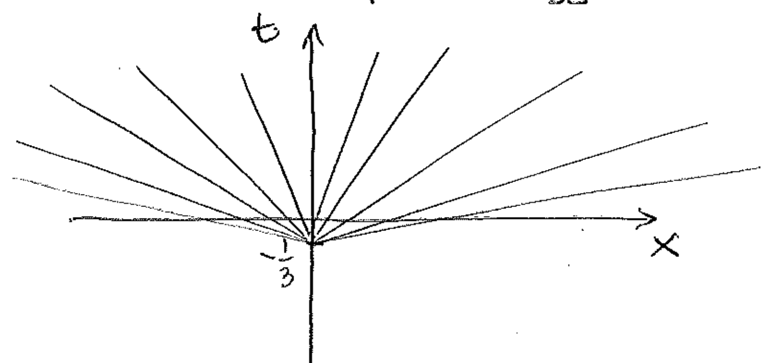
Interpretation

$u(x,0) = \phi(x) = 3x$



Characteristics are lines in the $(x\text{-}t)$ plane with slope $\frac{1}{F'(\phi(s))}$

Here the slope is $\frac{1}{3s}$



$$C^{(s)} : x = 3s \cdot t + s$$

$$\Leftrightarrow \quad t = \frac{x - s}{3s}$$

(they all pass through the point $(0, -\frac{1}{3})$)

$u(x,t)$ is constant along a characteristic

(2) Suppose $\phi(x) = e^{-x^2/2}$

then we have to invert

$$u(x,t) = e^{-(x - u(x,t)t)^2/2}$$

to find $u(x,t)$ → difficult.