2.2.4 Method of characteristics for quasilinear equations

General form: in (x, y) space

QL equations can be written as

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$$

- The equations determining the characteristics are similar to the semilinear case:

$$\begin{cases} \dfrac{\partial x}{\partial \tau} = a(x,y,u) \\ \dfrac{\partial y}{\partial \tau} = b(x,y,u) \\ \dfrac{\partial u}{\partial \tau} = c(x,y,u) \end{cases}$$

such that

$$\left(\frac{\partial x}{\partial \tau}\right)_s \frac{\partial u}{\partial x} + \left(\frac{\partial y}{\partial \tau}\right)_s \frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial \tau}\right)_s$$

- Note, however, that now the equations for $x^{(s)}(\tau)$ and $y^{(s)}(\tau)$ depend on the value of the function u itself so the system of ODEs is fully coupled.

Recall:

Semilinear case

$$\begin{cases} \dfrac{\partial x}{\partial \tau} = a(x,y) \\ \dfrac{\partial y}{\partial \tau} = b(x,y) \end{cases}$$

$$\frac{\partial u}{\partial \tau} = c(x,y,u)$$

$\Rightarrow$ The characteristics are independent of the value of the function u, and notably independent of $u_0(s)$ (of the initial condition)

The system decouples.

This time, the characteristics depend on the initial condition $u_0(s)$ of the system.

Definition:

- The characteristic curves are the 3D solutions of the system

$$\begin{cases} \frac{\partial x}{\partial \tau} = a(x, y, u) \\ \frac{\partial y}{\partial \tau} = b(x,y,u) \\ \frac{\partial u}{\partial \tau} = c(x,y,u) \end{cases}$$

and are parametrized as $\mathcal{C}^{(s)} = \begin{pmatrix} x^{(s)} \\ y^{(s)} \\ u^{(s)} \end{pmatrix}$

- The characteristics are the projection of the characteristic curves onto the (x,y) plane. They are parametrized as $\gamma^{(s)} = \begin{pmatrix} x^{(s)} \\ y^{(s)} \end{pmatrix}$

- For semilinear problems, characteristics can be calculated first, while $u^{(s)}(\tau)$ is calculated later to determine the solution

- In quasilinear problems, the characteristics cannot be calculated directly $\to$ the system is solved for the characteristic curves. $\begin{pmatrix} x^{(s)}(\tau) \\ y^{(s)}(\tau) \\ u^{(s)}(\tau) \end{pmatrix}$

The method is otherwise similar.

Example 1

$$\begin{cases} x \; u_x - u\, u_y = y \\ u(1,y) = y \end{cases}$$

① Initial condition curve

Let $\begin{cases} x_0(s) = 1 \\ y_0(s) = s \\ u_0(s) = s \end{cases}$ then $u(x_0(s), y_0(s)) = u(1,s) = s$

② Characteristic curves:

$$\begin{cases} \dfrac{dx}{\partial \tau} = x & \Rightarrow \quad x = x_0(s) e^{\tau} \\ \dfrac{\partial y}{\partial \tau} = -u \\ \dfrac{\partial u}{\partial \tau} = y \end{cases}$$

} a system of two coupled ODEs. Combine these to get

$$\frac{\partial^2 y}{\partial \tau^2} = -\frac{\partial u}{\partial \tau} = -y$$

So $\begin{cases} y = A \sin \tau + B \cos \tau \\ u = -\dfrac{\partial y}{\partial \tau} = -A \cos \tau + B \sin \tau \end{cases}$

Apply initial conditions

$$\begin{cases} x = e^{\tau} \\ y = -s \sin \tau + s \cos \tau = s(\cos \tau - \sin \tau) \\ u = s \cos \tau + s \sin \tau = s(\cos \tau + \sin \tau) \end{cases}$$

So $\tau = \ln x$ and $s = \dfrac{y}{\cos \tau - \sin \tau} = \dfrac{y}{\cos(\ln x) - \sin(\ln x)}$

So $u = \dfrac{y\,(\cos(\ln x) + \sin(\ln x))}{\cos(\ln x) - \sin(\ln x)}$

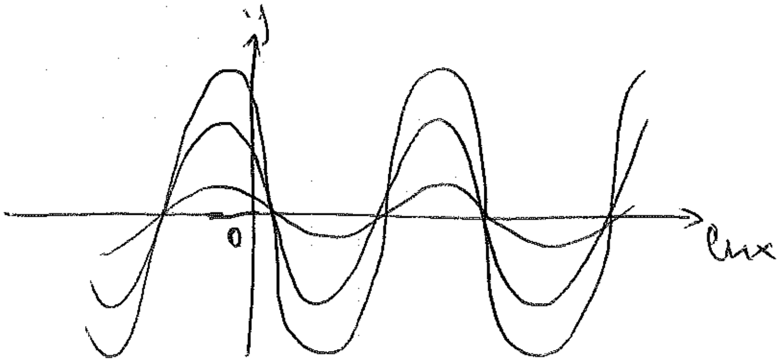
$$u(x,y) = y \, \frac{1 + \tan(\ln x)}{1 - \tan(\ln x)}$$

Question: ① What do the characteristics look like in (x,y) plane

② Where is the solution defined?

① $y = s\left(\cos(\ln x) - \sin(\ln x)\right)$

So naturally y is an oscillatory function of $\ln x$ with amplitude ranging from $-s\sqrt{2}$ to $+s\sqrt{2}$



zeros are at

$\ln x = \frac{\pi}{4} + k\pi$

$(x = e^{\frac{\pi}{4}+k\pi}$

→ Naturally, all characteristics cross at points

$$\begin{cases} x = e^{\frac{\pi}{4}+k\pi} \\ y = 0 \end{cases}$$

② When characteristics cross, the system

$\begin{cases} x(s,\tau) \\ y(s,\tau) \end{cases}$ is not invertible into $\begin{cases} s(x,y) \\ \tau(x,y) \end{cases}$

→ the solution is defined for

$$e^{-\frac{\pi}{4}} < x < e^{\pi/4}$$

but not outside of that interval

This corresponds to $u(x,y) = y\,\dfrac{1+\tan(\ln x)}{1-\tan(\ln x)}$

with the requirement

$$\tan(\ln x) \neq 1.$$

Example 2:
$$\begin{cases} \text{Same PDE with} \\ u(1, y) = -y \end{cases}$$

→ initial condition is slightly different (same position on the (x–y) plane, but a different value for u)

$$\begin{cases} x_0(s) = 1 \\ y_0(s) = s \\ u_0(s) = -s \end{cases}$$

→ Only difference is that

$$\begin{cases} x = e^{\tau} \\ y = s(\sin\tau + \cos\tau) \\ u = s(\sin\tau - \cos\tau) \end{cases}$$

so the characteristics are now

$$y = s\left(\sin(\ln x) + \cos(\ln x)\right) \Rightarrow \text{different from previous case}$$

This is a specific property of quasilinear equations vs semilinear equations: the characteristics are <u>not</u> uniquely defined by the PDE but also by the initial conditions. This effect is a consequence of the nonlinearity of the problem.

2.3 Existence and uniqueness

2.3.1 Introduction

- We are finding that the existence of a solution is associated with the invertibility of the mapping between the (s,τ) space and the (x,y) space.
- In some examples (see previously), this implied that the solution was only defined in a subset of $\mathbb{R}^2$.
- Can worse situations happen? Yes!

Let's compare two examples

$$\underline{\text{PDE 1}}: \quad x u_x + (x+y) u_y = u+1$$

$$\underline{\text{PDE2}}: \quad x u_x + y u_y = u+1$$

with initial condition

$$u(x,x) = x^2 \quad : \quad \begin{cases} x_0 = s \\ y_0 = s \\ u_0 = s^2 \end{cases}$$

Case 1. Integrate

$$\begin{cases} \dfrac{\partial x}{\partial \tau} = x & \Rightarrow \quad x = x_0(s) e^{\tau} \\ \dfrac{\partial y}{\partial \tau} = x+y & \Rightarrow \quad \dfrac{\partial y}{\partial \tau} = x_0(s) e^{\tau} + y \\ \dfrac{\partial u}{\partial \tau} = u+1 & \Rightarrow \quad u = \left(u_0(s)+1\right) e^{\tau} - 1 \end{cases}$$

To solve for y, use an integrating factor method (for example),

$$\frac{dy}{d\tau} - y = x_0(s) e^{\tau}$$

so $\mu = e^{-\tau}$ and

$$\frac{d}{d\tau}\left(ye^{-\tau}\right) = x_0(s)$$

$$\rightarrow \quad ye^{-\tau} = c + x_0(s)\tau$$

so $\quad y = ce^{\tau} + x_0(s)\tau e^{\tau}$

To ensure $y = y_0(s)$ when $\tau = 0$ choose

$$y = y_0(s)e^{\tau} + x_0(s)\tau e^{\tau}$$

So finally
$$\begin{cases} x = se^{\tau} \\ y = se^{\tau}(\tau+1) \\ u = (s^2+1)e^{\tau} - 1 \end{cases}$$

so $\quad \frac{y}{x} = \tau + 1 \Rightarrow \tau = \frac{y}{x} - 1$

so $\quad s = xe^{-\tau} = xe^{-\left(\frac{y}{x}-1\right)}$

and therefore

$$u = \left[x^2 e^{-2\left(\frac{y}{x}-1\right)} + 1\right] e^{\frac{y}{x}-1} - 1$$

$\Rightarrow$ provided $x \neq 0$ no problem here.

Case 2: The characteristics are obtained by integrating

$$\begin{cases} \frac{\partial x}{\partial \tau} = x & \rightarrow & x = x_0(s)e^{\tau} = se^{\tau} \\ \frac{\partial y}{\partial \tau} = y & \rightarrow & y = y_0(s)e^{\tau} = se^{\tau} \\ \frac{\partial u}{\partial \tau} = u+1 & \rightarrow & u = \left[s^2+1\right]e^{\tau} - 1 \end{cases}$$

Similarly, we try to invert the mapping: we find

$$\frac{y}{x} = 1$$

$\rightarrow$ the mapping $\begin{cases} x = se^{\tau} \\ y = se^{\tau} \end{cases}$ only maps the $x = y$ line!

$\Rightarrow$ $u(s,\tau)$ exists for all s and τ but we cannot invert the mapping for x and y

Moreover, the initial condition $u(x,x) = x^2$ does not satisfy the PDE on Γ :

$$\begin{aligned} & x u_x + y u_y \\ = \; & x(2x) + x(2x) = 4x^2 \neq x^2 \end{aligned}$$

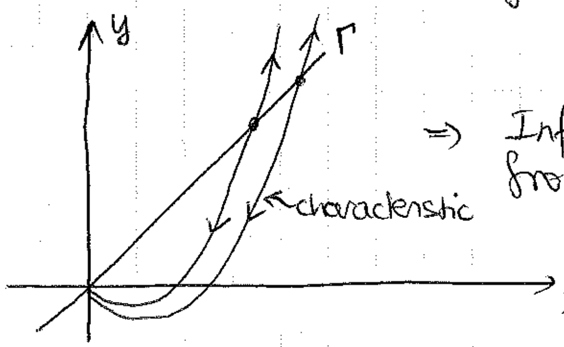
$\Rightarrow$ No solutions to this problem.

What is $\neq$ between these two cases in terms of the characteristics of the system?

<u>Case 1</u> The characteristics are given by $x = s e^{\frac{y}{x}+1}$

or equivalently $y = x \left[\ln\left(\frac{x}{s}\right) - 1 \right]$

$\rightarrow$



$\Rightarrow$ Information is transported away from Γ on characteristics.

$\Rightarrow$ No problem until the characteristic reaches $x = 0$; there, the mapping is not invertible

<u>Case 2</u> The characteristics are y x which is <u>also</u> the equation for the initial condition curve.

$\Rightarrow$ the initial information cannot be transported away from Γ.