Back to the first order PDE

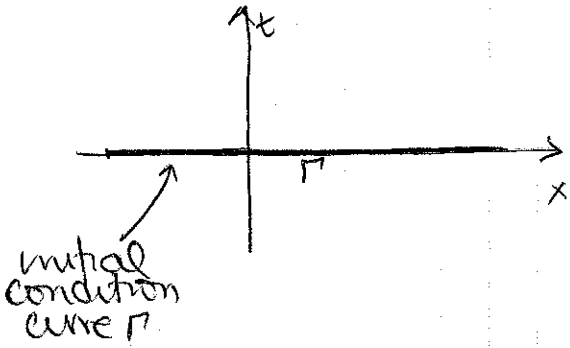
Step 1: We represent the additional condition curve as a parametric curve with parameter s

Suppose we know $u(x,t)$ on a particular curve Γ in the (x,t) plane. Let's parametrize Γ with the functions $x_0(s)$, $t_0(s)$ such that

$$\Gamma = \begin{cases} x_0(s) \\ t_0(s) \end{cases}$$

then on this curve $u(x_0(s), t_0(s)) = u_0(s)$

Examples: • Suppose we want to impose $u(x, 0) = 3x$



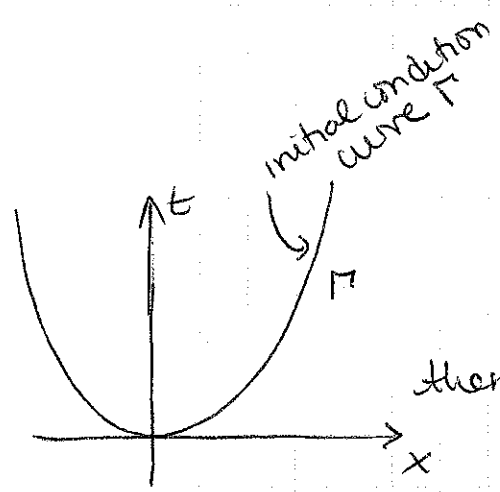
The initial condition curve has $t=0$ for all x $\rightarrow$

Parametrize it (for example) as

$$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \end{cases} \Rightarrow u_0(s) = 3s$$

or we could also use

$$\begin{cases} x_0(s) = s^2 \\ t_0(s) = 0 \end{cases} \Rightarrow u_0(s) = 3s^2$$



• Suppose $u(x, x^2) = e^x$

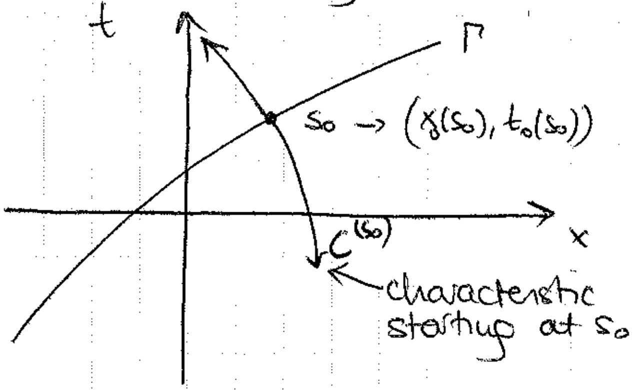
then $$\begin{cases} x_0(s) = s \\ t_0(s) = s^2 \end{cases} \Rightarrow u_0(s) = e^s$$

Note: Since there are many possible parametric representations of the same curve, always try to choose the simplest one:

Prefer $$\begin{cases} x_0(s) = s \\ t_0(s) = s^2 \end{cases}$$ over $$\begin{cases} x_0(s) = \ln(s^2) \\ t_0(s) = [\ln(s^2)]^2 \end{cases}$$!

Step 2: Now that we have parametrized the "initial" condition curve we want to identify special curves along which the PDE behaves as an ODE. These are called characteristics.

① Suppose that for a selected point on the initial curve there exists only <u>one</u> characteristic emanating from it



$\Rightarrow$ let's parametrize this characteristic with the new parameter τ

$$C^{(s_0)} = \begin{cases} x^{(s_0)}(\tau) \\ t^{(s_0)}(\tau) \end{cases}$$

$\Rightarrow$ on this curve

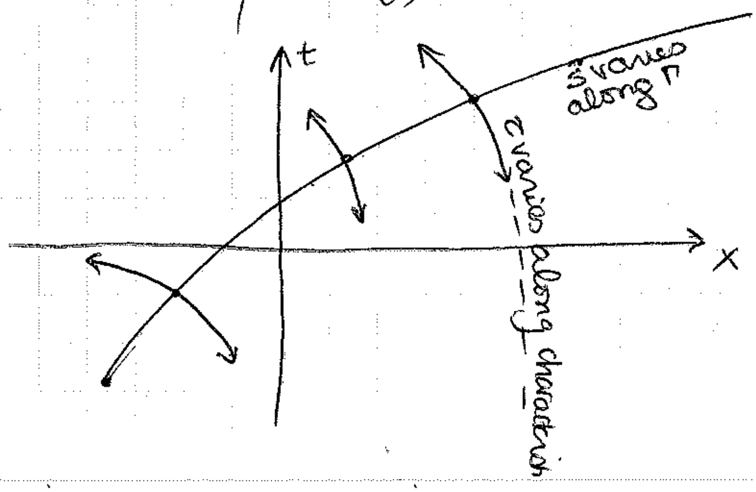
$$u\left(x^{(s_0)}(\tau), t^{(s_0)}(\tau)\right) = u^{(s_0)}(\tau)$$

u depends on τ only

② Now let's do this same construction for every point $[x_0(s), t_0(s)]$ on the initial condition curve

$\Rightarrow$ We get a family of characteristics, each starting from a point identified with the parameter s, and each parametrized with τ:

$$C^{(s)} \begin{cases} x^{(s)}(\tau) \\ t^{(s)}(\tau) \end{cases} \quad \text{with } u^{(s)}(\tau) = u\left(x^{(s)}(\tau), t^{(s)}(\tau)\right)$$



$\Rightarrow$ Note that what we have really done, is to remap the (x,t) space onto the (s,τ) space

So that the function

$$u(x,t) \text{ is also } u\left(x^{(s)}(\tau), t^{(s)}(\tau)\right)$$

$$= u(s,\tau)$$

with the added requirement that the PDE

$$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} \quad \text{behaves like an ODE}$$

when restricted to a characteristic $(s = \text{constant})$.

How do we impose this requirement?

Note that $\left.\frac{\partial u}{\partial \tau}\right)_s$ is the derivative of u along a characteristic. (i.e holding s constant)

derivative of u w.r.t parameter τ at constant s

By multivariate chain rule, and using $\begin{pmatrix} x^{(s)}(\tau) \\ t^{(s)}(\tau) \end{pmatrix}$ on characteristic

$$\left.\frac{\partial u}{\partial \tau}\right)_s = \frac{\partial u}{\partial x}\left.\frac{\partial x}{\partial \tau}\right)_s + \frac{\partial u}{\partial t}\left.\frac{\partial t}{\partial \tau}\right)_s$$

$$= \frac{\partial u}{\partial x}\frac{d}{d\tau}\left[x^{(s)}\right] + \frac{\partial u}{\partial t}\frac{d t^{(s)}}{d\tau}$$

Going back to the original PDE, if

$$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = \frac{d}{d\tau}\left[t^{(s)}\right]\frac{\partial u}{\partial t} + \frac{d}{d\tau}\left[x^{(s)}\right]\frac{\partial u}{\partial x}$$

then $\Rightarrow \left.\frac{\partial u}{\partial \tau}\right)_s = c_1 u + c_0$

Now the PDE looks like an ODE for s held constant, i.e, along a characteristic.

This occurs when $\boxed{\frac{dt^{(s)}}{d\tau} = a \qquad \frac{dx^{(s)}}{d\tau} = b}$

$\Rightarrow$

$$C^{(s)} = \begin{cases} t^{(s)} = a\tau + \text{constant specific to this characteristic} \\ x^{(s)} = b\tau + \text{constant} \quad " \quad " \quad " \quad " \end{cases}$$

Suppose we require that when $\tau = 0$ we are on the initial condition curve then at $\tau = 0$

$$t^{(s)} = t_0(s)$$
$$x^{(s)} = x_0(s)$$

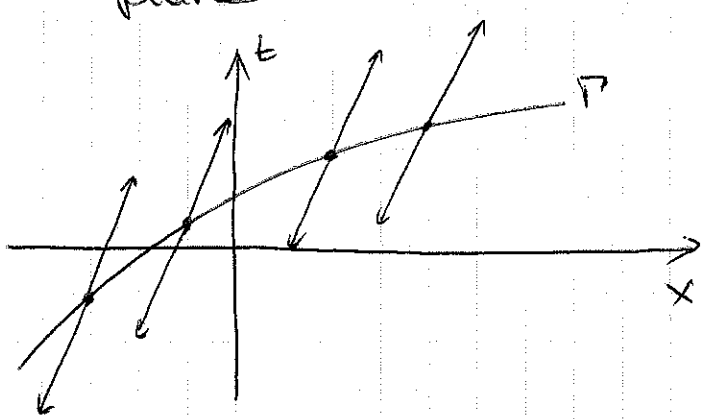
thus

$$C^{(s)} = \begin{cases} t^{(s)} = a\tau + t_0(s) \\ x^{(s)} = b\tau + x_0(s) \end{cases}$$

is the parametric equation for the characteristic emanating from the point $[x_0(s), t_0(s)]$ on the initial condition curve.

<u>What do these characteristics look like?</u>

Here, they are straight lines with slope $\frac{a}{b}$ in the (x,t) plane



<u>Note</u>: this is only true when the PDE has constant coefficients (see later)

<u>Step 3</u> <u>What is the solution to the PDE?</u>

Now we have to solve $\frac{\partial u}{\partial \tau} = c_1 u + c_0$ for each s

$$\Rightarrow \quad u = A(s) e^{c_1\tau} - \frac{c_0}{c_1} + \frac{c_0}{c_1} e^{c_1\tau} \qquad \text{(check this)}$$

where the arbitrary constant $A(s)$ is chosen such that $u = u_0(s)$ when $\tau = 0$:

$$u(s,\tau) = u_0(s) e^{c_1\tau} - \frac{c_0}{c_1} + \frac{c_0}{c_1} e^{c_1\tau}$$

How do we get a solution in terms of (x,t)?

Invert the system (if possible)

$$\begin{cases} t = a\tau + t_0(s) \\ x = b\tau + x_0(s) \end{cases}$$

to write τ and s in terms of x and t, then plug into $u(s,\tau)$

Example 1 Suppose we want to solve the simple transport equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + V_0 \dfrac{\partial u}{\partial x} = 0 \\ u(x,0) = e^{-x^2/2} \end{cases} \qquad \text{(a Gaussian)}$$

Step 1: Parametrize the initial condition

$$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = e^{-s^2/2} \end{cases}$$

Step 2 The characteristic curves are such that

$$\begin{pmatrix} a=1 \\ b=V_0 \end{pmatrix} \qquad \begin{cases} \dfrac{\partial t^{(s)}}{\partial \tau} = 1 \\ \dfrac{\partial x^{(s)}}{\partial \tau} = V_0 \end{cases} \Rightarrow \begin{cases} t^{(s)} = \tau + t_0(s) \\ x^{(s)} = V_0\tau + x_0(s) \end{cases} \Rightarrow \begin{cases} t^{(s)} = \tau \\ x^{(s)} = V_0\tau + s \end{cases}$$

Step 3: The solution to $\dfrac{\partial u}{\partial \tau} = 0$ $\quad (c_1 = c_0 = 0)$

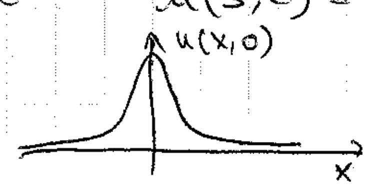
is $u =$ constant on a characteristic

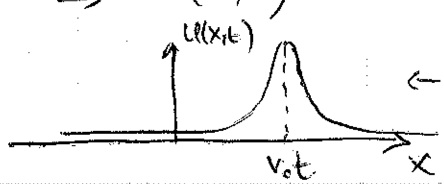
$$\Rightarrow u = u_0(s) = e^{-s^2/2}$$

Step 4:

$$\begin{cases} t = \tau \\ x = V_0\tau + s \end{cases} \Rightarrow \begin{cases} \tau = t \\ s = x - V_0 t \end{cases}$$

so $u(s,\tau) = e^{-s^2/2} \Rightarrow u(x,t) = e^{-(x-V_0t)^2/2}$





← a "travelling" Gaussian i.e horizontally moved.

Step 5 Always check your answer

$$\frac{\partial u}{\partial t} = - v_0 \, (x - v_0 t) \, e^{-(x-v_0t)^2/2}$$

$$\frac{\partial u}{\partial x} = (x - v_0 t) \, e^{-\frac{(x-v_0t)^2}{2}}$$

$$\frac{\partial u}{\partial t} + v_0 \frac{\partial u}{\partial x} = 0 \quad \checkmark$$

Example 2 : General case

$$\begin{cases} \dfrac{\partial u}{\partial t} + v_0 \dfrac{\partial u}{\partial x} = 0 \\ u(x,0) = F(x) \end{cases}$$

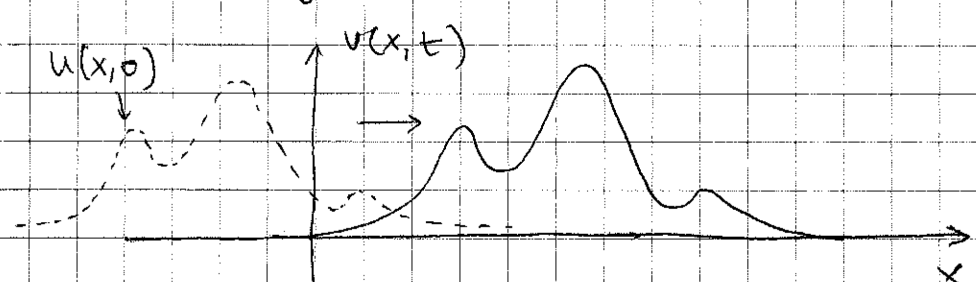
Step 1 :
$$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = F(s) \end{cases}$$

Step 2 :
$$\begin{cases} \dfrac{dt}{d\tau} = 1 \\ \dfrac{dx}{d\tau} = v_0 \end{cases} \Rightarrow \begin{cases} t = \tau \\ x = v_0 \tau + s \end{cases}$$

Step 3
$$\frac{du}{d\tau} = 0 \quad \Rightarrow \quad u = u_0(s) = F(s)$$

Step 4 :
$$\begin{cases} \tau = t \\ s = x - v_0 t \end{cases} \rightarrow \boxed{u(x,t) = F(x - v_0 t)}$$

$\Rightarrow$ The initial condition $F(x)$ "moves" with velocity v_0



Example 3 :

$$\begin{cases} \frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = 0 \\ u(x,0) = e^{-x^2} \end{cases}$$

Step 1 :

$$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = e^{-s^2} \end{cases}$$

Step 2

$$\begin{cases} \frac{dt}{d\tau} = 1 \\ \frac{dx}{d\tau} = x \end{cases} \Rightarrow \begin{cases} t = \tau + t_0(s) = \tau \\ x = x_0(s)e^{\tau} = se^{\tau} \end{cases}$$

Step 3

$$\frac{du}{d\tau} = 0 \Rightarrow u = u_0(s) = e^{-s^2}$$

Step 4 :

$$\begin{cases} \tau = t \\ s = xe^{-t} \end{cases} \Rightarrow u(x,t) = e^{-(xe^{-t})^2} = e^{-\frac{x^2}{e^{2t}}}$$

This describes a Gaussian with constant amplitude 1, constant mean, but with a width which grows exponentially in time .

2.3.3 Semilinear equations

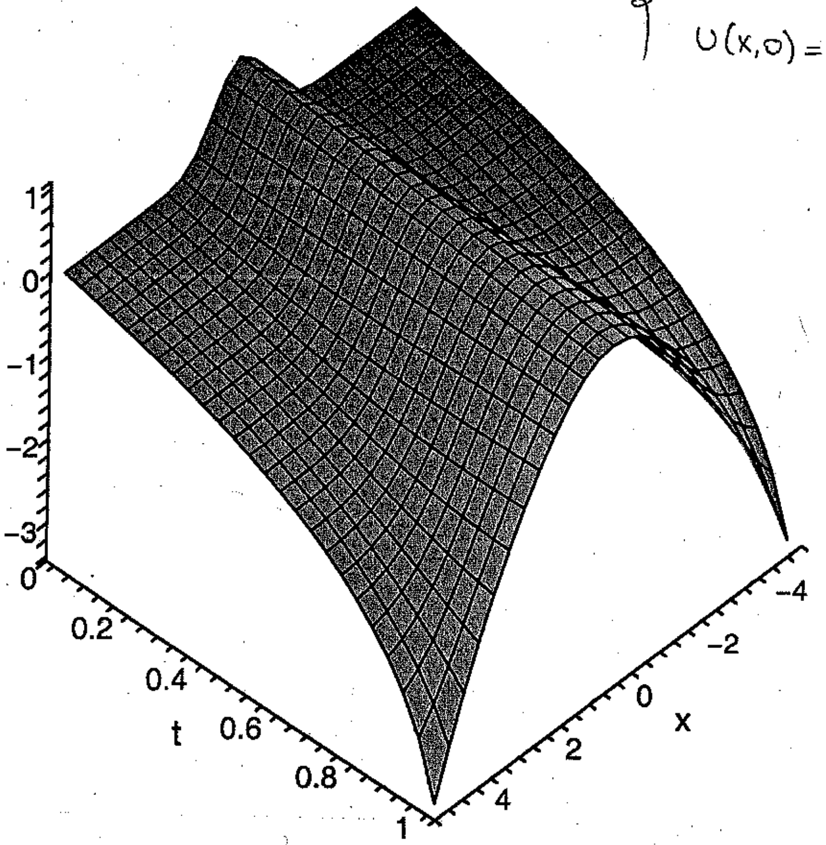
The method for semilinear equations :

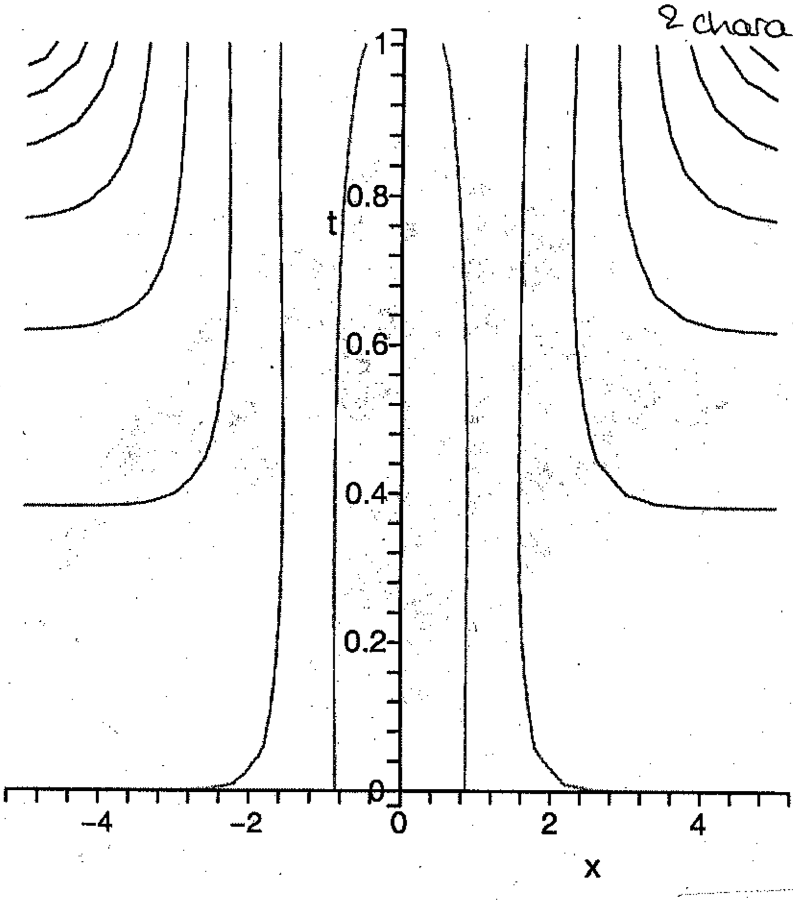
$$\begin{cases} a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t,u) \\ u(x,0) = \phi(x) \end{cases}$$

is the same as for linear equations. However, note that the resulting ODE for u will be nonlinear.

Solution 2 Contour plot for

$$\begin{cases} u_t + x u_x = -e^{-u} \\ u(x,0) = e^{-x^2} \end{cases}$$





Note that contour lines & characteristics do <u>not</u> coincide.

__Example__
$$\begin{cases} \dfrac{\partial u}{\partial t} + x\dfrac{\partial u}{\partial x} = -e^{-u} \\ u(x,0) = e^{-x^2} \end{cases}$$

__Step 1__

$$\Gamma \begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = e^{-s^2} \end{cases}$$

__Step 2__

$$\begin{cases} \dfrac{dt}{d\tau} = 1 \\ \dfrac{dx}{d\tau} = x \end{cases} \Rightarrow \begin{cases} t = \tau + t_0(s) = \tau \\ x = x_0(s)e^{\tau} = se^{\tau} \end{cases}$$

__Step 3__

$$\frac{du}{d\tau} = -e^{-u} \Rightarrow \frac{du}{e^{-u}} = -d\tau \Rightarrow e^{u}du = -d\tau$$

$$e^{u} = -\tau + K(s) \Rightarrow u = \ln(\tau + K(s))$$

$$\text{At} \quad \tau = 0 \quad e^{u_0(s)} = K(s) \Rightarrow K(s) = e^{e^{-s^2}}$$

$$u = \ln\left(-\tau + e^{e^{-s^2}}\right)$$

__Step 4__

$$\begin{cases} \tau = t \\ s = xe^{-t} \end{cases} \Rightarrow u(x,t) = \ln\left(-t + e^{e^{-x^2e^{-2t}}}\right)$$

__Step 5__ Check:

$$\frac{\partial u}{\partial t} = \frac{-1 + 2x^2e^{-2t}e^{-x^2e^{-2t}}e^{e^{-x^2e^{-2t}}}}{-t + e^{e^{-x^2e^{-2t}}}}$$

$$\frac{\partial u}{\partial x} = -\frac{2xe^{-2t}e^{-x^2e^{-2t}}e^{e^{-x^2e^{-2t}}}}{-t + e^{e^{-x^2e^{-2t}}}}$$

so ✓.

What did we learn?

① Method of solution of semilinear, first order PDEs

Step 1: Parametrize the initial condition curve

Step 2: If $a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t,u)$

then the characteristics are found by solving the system

$$\left\{\begin{array}{l} \frac{\partial t^{(s)}}{\partial \tau} = a(x,t) \\ \frac{\partial x^{(s)}}{\partial \tau} = b(x,t) \end{array}\right\}$$

Note that these are coupled ODEs.

with the initial condition

$$t^{(s)}(\tau=0) = t_0(s)$$
$$x^{(s)}(\tau=0) = x_0(s)$$

Step 3: The solution to the PDE in (s,τ) is found by solving

$$\frac{\partial u^{(s)}}{\partial \tau} = c(x,t,u)$$

(note that x and t depend on s and τ)

Step 4: If possible, invert the system

$$\left\{\begin{array}{l} t(s,\tau) \\ x(s,\tau) \end{array}\right. \quad \text{to get} \quad \left\{\begin{array}{l} \tau(x,t) \\ s(x,t) \end{array}\right.$$

and plug into $u(s,\tau)$ to get $u(x,t)$.

Step 5 Check answer.

② Note:

- When the linear PDE is homogeneous ($c(x,t) = 0$) then

$$\frac{\partial u}{\partial \tau} = 0 \Rightarrow$$ u is constant along characteristics; in other words, the characteristics are contour levels of the solution $u(x,t)$.

- When the PDE is not homogeneous then u is not constant along characteristics. The characteristics propagate the initial condition according to the equation

$$\frac{\partial u^{(s)}}{\partial \tau} = c\left(x^{(s)}(\tau), t^{(s)}(\tau), u^{(s)}(\tau)\right)$$

(see examples later)

③ Question.

- What is the condition for the mapping
$$\begin{cases} \underset{\sim}{x}(s,\tau) \\ t(s,\tau) \end{cases}$$ to be invertible?

- What happens if the characteristics are somewhere parallel to the initial condition curve?