6.2.4 Integral solution for the Dirichlet problem for the Poisson equation

Consider the Dirichlet Poisson problem $\begin{cases} \nabla^2 u = f & (x,y)\in D \\ u = g & (x,y)\in \partial D \end{cases}$

Given the expression

$$u(\xi,\eta) = \int_{\partial D} (\Gamma \partial_n u - u \partial_n \Gamma)\, ds - \int_D \Gamma \nabla^2 u \, dV$$

we know u on ∂D $(u = g)$
$\nabla^2 u$ in D so $(\nabla^2 u = f)$.

$$u(\xi,\eta) = \int_{\partial D} \Gamma \partial_n u - g u \, ds - \int \Gamma f \, dV$$

Problem: we do not know $\partial_n u$, so we cannot use this directly.

Idea: Consider the function

$$G(x,y;\xi,\eta) = \Gamma(x,y;\xi,\eta) - h(x,y;\xi,\eta)$$

where $h(x,y;\xi,\eta)$ is the solution to

$$\begin{cases} \nabla^2 h = 0 & \text{in } D \\ h = \Gamma & \text{on } \partial D. \end{cases}$$

then $\nabla^2 G = -\delta(x-\xi, y-\eta)$ by construction and
$G = 0$ on ∂D so that using
Green's # 2 identity we now have

$$\int_D u\nabla^2 G - G\nabla^2 u \, dxdy = \int_{\partial D} (u\partial_n G - G\partial_n u)\, ds$$

$$\Leftrightarrow \quad \left. \begin{aligned} u(\xi,\eta) = &-\int_D G(x,y;\xi,\eta) f(x,y)\, dxdy \\ &- \int_{\partial D} g(x,y)\, \partial_n G \, ds \end{aligned} \right\} \text{an integral representation of the solution using } \underline{\text{Green's Function } G}.$$

Note This doesn't quite solve the problem yet since we still have to find h. However, there are tricks to find h for simple geometries (see later).

6.2.5 Integral solution of the Poisson equation for the Neumann problem

Consider
$$\begin{cases} \nabla^2 u = f & (x,y) \in D \\ \partial_n u = g & (x,y) \in \partial D \end{cases}$$

in the expression
$$u(\xi,\eta) = \int_{\partial D} \Gamma \partial_n u - u \partial_n \Gamma \, ds - \int_D \Gamma \nabla^2 u \, dV$$

we know $\partial_n u$ and $\nabla^2 u$ but not u on ∂D.

This time consider the function
$$N(x,y;\xi,\eta) = \Gamma(x,y,\xi,\eta) - h(x,y;\xi,\eta)$$

with $\begin{cases} \nabla^2 h = 0 \\ \partial_n h = \partial_n \Gamma + \frac{1}{L} \end{cases}$ where L is the length of ∂D.

Again $\nabla^2 N = -\delta(x,y;\xi,\eta)$ by construction

and $\partial_n N = \partial_n \Gamma - \partial_n \Gamma - \frac{1}{L} = -\frac{1}{L}$

Note : For $\nabla^2 h = 0$ and $\partial_n h = \partial_n \Gamma + \frac{1}{L}$ to have a solution we must verify that
$\int_{\partial D} \partial_n h \, ds = 0$. This is true because
$\int_{\partial D} \partial_n \Gamma \, ds = -1$ and $\int_{\partial D} \frac{1}{L} \, ds = 1$. (*)

Using Green's # 2 identity we get

$$\int u\nabla^2 N - N\nabla^2 u \, dV = \int_{\partial D} u\partial_n N - N\partial_n u \, ds$$

$$-u(\xi,\eta) = \int_V N\nabla^2 u - \int_{\partial D} Ng \, ds - \int_{\partial D} \frac{u}{L} \, ds$$

$$\boxed{u(\xi,\eta) = -\int_V Nf + \int_{\partial D} Ng \, ds + \int_{\partial D} \frac{u}{L} \, ds}$$

Note: N is defined to within an additive constant $\int_{\partial D} \frac{u}{L} ds$

(*) : to prove $\int_{\partial D} \partial_n \Gamma \, ds = -1$ simply use

$$u(\xi,\eta) = \int_{\partial D} \Gamma\partial_n u - u\partial_n \Gamma - \int_D \Gamma\nabla^2 u \, ds$$

with $u \equiv 1$ (since it holds for any smooth function u)

then $1 = \int_{\partial D} 0 - \partial_n \Gamma \, ds - 0$

Alternatively recall that $\nabla^2 \Gamma$ is a δ function so that

$$\int_{\partial D} \partial_n \Gamma \, ds = \int_D \nabla^2 \Gamma \, dV = 1$$

Note: In any Neumann problem the solution is defined to within an additive constant. The term

$\int_{\partial D} \frac{u}{L} ds$ incorporates that constant.

(For instance, choose $\int_{\partial D} u \, ds = 0$ and $\int_{\partial D} N(x,y;\xi,\eta) \, ds = 0$ as normalisations, then the solution is uniquely defined).

6-3 Examples

(1) Consider the Dirichlet problem in the half-space

$$\nabla^2 \psi = f(x,y) \qquad (x > 0)$$
$$\psi = 0 \quad \text{for} \quad x = 0$$

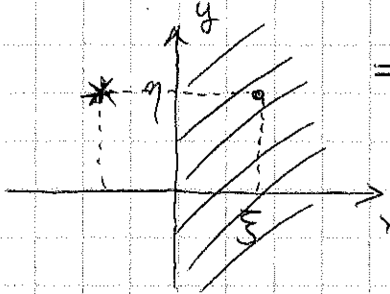
The fundamental solution to Laplace equation is $\Gamma(x,y;\xi,\eta) = -\frac{1}{2\pi}\ln\left((x-\xi)^2 + (y-\eta)^2\right)$

We now need to find solutions to

$$\begin{cases} \nabla^2 h = 0 & \text{for } x > 0 \\ h = \Gamma & \text{on } x = 0 \end{cases} \quad (\text{so } h(0,y;\xi,\eta) = \Gamma(0,y;\xi,\eta))$$

Method of images (reflection principle)

Idea: Construct the function h to be the symmetric function of Γ across the domain boundary:



$\Rightarrow$ If Γ has a pole in (ξ,η), construct a function that has a pole in $(-\xi,\eta)$

here $h = \Gamma(x,y;-\xi,\eta)$

so that $G = \Gamma(x,y;\xi,\eta) - \Gamma(x,y;-\xi,\eta)$

We can verify that

$\nabla^2 G = \nabla^2 \Gamma$ in D (since $\nabla^2 h = 0$ everywhere in D (the pole is outside of D))

$G(x=0) = 0$

since $h(0,y,\xi,\eta) = \Gamma(0,y;\xi,\eta)$

so the solution to the problem is

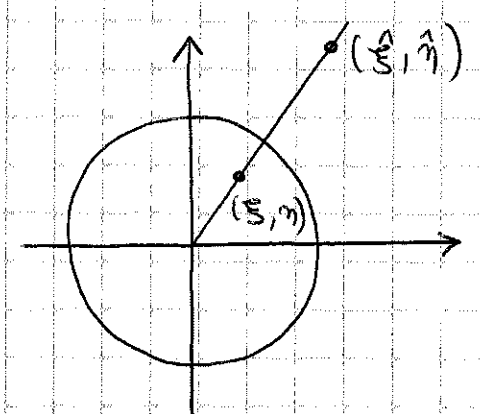
$$u(\xi,\eta) = -\int_{x>0} G(x,y,\xi,\eta)\, f(x,y)\, dx\,dy - \int_{x=0} 0 \cdot \partial_n G \, ds = -\int_{y=-\infty}^{+\infty} \int_{x=0}^{\infty} G(x,y,\xi,\eta) f(x,y) dx\,dy$$

② A "similar" construction can be used to compute h in a disk.

Consider the Dirichlet problem in the disk

$$\begin{cases} \nabla^2 \psi = f(x,y) & (x^2+y^2)^{1/2} < R \\ \psi(x,y) = g(x,y) & (x^2+y^2)^{1/2} = R \end{cases}$$

$\Rightarrow$ we want to find, for each (ξ,η), the function $h(x,y;\xi,\eta)$ such that



$$\begin{cases} \nabla^2 h = 0 & \text{in the disk} \\ h(x,y;\xi,\eta) = \Gamma(x,y;\xi,\eta) & \text{on the disk} \end{cases}$$

Trick: Consider the "inverse" point of (ξ,η), $(\hat{\xi},\hat{\eta})$ defined as

$$\begin{pmatrix} \hat{\xi} \\ \hat{\eta} \end{pmatrix} = \frac{R^2}{\xi^2+\eta^2} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

and let

$$h(x,y,\xi,\eta) = \Gamma\left[\frac{\sqrt{\xi^2+\eta^2}}{R} x, \frac{\sqrt{\xi^2+\eta^2}}{R} y; \frac{\sqrt{\xi^2+\eta^2}}{R} \hat{\xi}, \frac{\sqrt{\xi^2+\eta^2}}{R} \hat{\eta} \right]$$

$$= \Gamma\left[\frac{\sqrt{\xi^2+\eta^2}}{R} x, \frac{\sqrt{\xi^2+\eta^2}}{R} y; \frac{R}{\sqrt{\xi^2+\eta^2}} \xi, \frac{R}{\sqrt{\xi^2+\eta^2}} \eta \right]$$

Check:

(1) $$\nabla^2 h = h_{xx} + h_{yy} = \frac{\xi^2+\eta^2}{R^2}\,\nabla^2\Gamma = 0$$

(2) $$h(x,y;\xi,\eta) = -\frac{1}{2\pi}\ln\left[\frac{\xi^2+\eta^2}{R^2}(x-\hat{\xi})^2 + \frac{\xi^2+\eta^2}{R^2}(y-\hat{\eta})^2\right]$$

should be equal to

$$\Gamma(x,y;\xi,\eta) = -\frac{1}{2\pi}\ln\left[(x-\xi)^2+(y-\eta)^2\right]$$

on the circle $x^2+y^2=R^2$

$$\frac{\xi^2+\eta^2}{R^2}\left[(x-\hat{\xi})^2+(y-\hat{\eta})^2\right]$$
$$= \frac{\xi^2+\eta^2}{R^2}\left[x^2+y^2-2x\hat{\xi}-2y\hat{\eta}+\hat{\xi}^2+\hat{\eta}^2\right]$$
$$= (\xi^2+\eta^2) - (2x\xi+2y\eta) + \frac{R^2}{\xi^2+\eta^2}(\xi^2+\eta^2)$$
$$= R^2 - 2x\xi + 2y\eta + (\xi^2+\eta^2)$$
$$= (x-\xi)^2+(y-\eta)^2 \qquad \Box.$$

$\Rightarrow$ The Green's function on the disk of radius R is

$$G(x,y;\xi,\eta) = \Gamma(x,y;\xi,\eta) - \Gamma\left(\frac{\sqrt{\xi^2+\eta^2}}{R}x, \frac{\sqrt{\xi^2+\eta^2}}{R}y; \frac{R}{\sqrt{\xi^2+\eta^2}}\xi, \frac{R}{\sqrt{\xi^2+\eta^2}}\eta\right)$$
$$= -\frac{1}{2\pi}\ln\left((x-\xi)^2+(y-\eta)^2\right)$$
$$+\frac{1}{2\pi}\ln\left[\left(\frac{\sqrt{\xi^2+\eta^2}}{R}x - \frac{R}{\sqrt{\xi^2+\eta^2}}\xi\right)^2 + \left(\frac{\sqrt{\xi^2+\eta^2}}{R}y - \frac{R}{\sqrt{\xi^2+\eta^2}}\eta\right)^2\right]$$

and the solution to the Dirichlet problem

$$\begin{cases}\nabla^2 u = f & \text{in } D\\ u = g & \text{on } D\end{cases}$$

is $$u(\xi,\eta) = -\iint_D Gf\,dx\,dy - \int_{\partial D} g\frac{\partial G}{\partial r}\,d\ell$$

(r = radial coordinate)