II Elliptic equations. Green's functions

A Prototype elliptic problems

- Recall that elliptic equations can always be cast into their canonical form

$$u_{xx} + u_{yy} + \mathcal{L}^{(1)}(u) = 0$$

where $\mathcal{L}^{(1)}(u)$ is a linear operator acting on u.

- Therefore, the simplest equations considered are
  - the Laplace equation
  $$u_{xx} + u_{yy} \left(= \nabla^2 u\right) = 0$$
  - the Poisson equation
  $$u_{xx} + u_{yy} = F(x,y)$$

A.1 Divergence theorem & consequences for elliptic problems

- Recall: the divergence theorem states that

$$\int_V \nabla \cdot \underline{u} \, dV = \int_{\partial V} \underline{u} \cdot \underline{n} \, dS$$

$\underline{n}$ = normal unit vector to the "surface" ∂V.

V can be here a volume/surface, then ∂V is a surface/contour.

- $\nabla^2 u = \nabla \cdot (\nabla u)$

cf. in cartesian:
$$\nabla u = \begin{pmatrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial u/\partial z \end{pmatrix}$$

so
$$\nabla \cdot \nabla u = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)$$
$$= \nabla^2 u .$$

As a result, for an elliptic Poisson-type PDE we see that

$$\int_V \nabla^2 u \, dV = \int_{\partial V} \nabla u \cdot \underline{n} \, dS = \int_V F(\underline{r}) \, dV.$$

$\Rightarrow$ the nature of the equation provides a constraint on the solution at the boundaries.

<u>Consequence</u>: recall that for Neumann-type boundary conditions we impose $\nabla \underline{u} \cdot \underline{n}$ $(= \partial_n u)$ on the domain contour

$\Rightarrow$ only when $\int_{\partial V} \partial_n u \, . \, dS = \int_V F(r) dV$ (*)

will the Neumann problem have a solution.

<u>In general</u>:
- the problem of existence of solutions for elliptic equations is much more complex than for parabolic / hyperbolic equations.
- Provided the domain considered is <u>bounded</u> and <u>smooth enough</u>.

see later on

- solutions to the Dirichlet problem exist and are unique
- solutions to the Neumann problem exist (if * holds) but are not unique ($u = v + K$, $K \in \mathbb{R}$ is also solution)
- solutions to the Robin problem exist and are unique.

A.2 Harmonic functions

Definition: a harmonic function $u(x,y)$ in a domain D is a function satisfying
$$\nabla^2 u = 0 \quad \text{for all } (x,y) \in D$$

Examples: (among others).

1. Let's look for polynomial solutions:

e.g. 2nd order: (quadratic form)
$$ax^2 + bxy + cy^2 = 0$$
$$\Rightarrow \quad 2a + 2c = 0$$
$\Rightarrow$ • $a = -c$
• b can be any value

So any function of the kind
$$A(x^2 - y^2) + Bxy + \alpha x + \beta y + \gamma = 0$$
is a harmonic function in the whole plane.

2. Let's look for center symmetric solutions

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0$$
$$\Rightarrow \quad r\frac{\partial u}{\partial r} = K \quad \Rightarrow \quad \frac{\partial u}{\partial r} = \frac{K}{r} \quad \Rightarrow$$
$$u = +K\ln r + K'.$$

Note that $u(r)$ is <u>not</u> defined at $r=0$.
$\rightarrow$ harmonic in $\mathbb{R}^2 - \{0\}$

In cartesians:
$$u(x,y) = \frac{K}{2}\ln(x^2+y^2) + K'.$$

1.3 The maximum principles and mean value principle

① <u>Theorem</u>: <u>The weak maximum principle</u>

Let D be a <u>bounded</u> domain, and $u(x,y)$ a function continuous & differentiable in D satisfying $\nabla^2 u = 0$ in D (a harmonic function)

Then the maximum of u is achieved <u>on</u> the boundary ∂D

② <u>Corollary</u>: the above theorem also holds for the minimum of u, because

- $v = -u$ is also a harmonic function and
- $\max(v) = \min(u)$

Idea behind the theorem (see Textbook)

For any local maximum <u>within</u> D, we necessarily have $\nabla^2 u \leq 0$
(recall, for 1D functions, x_0 is a local maximum of u $\Rightarrow$ $u''(x_0) \leq 0$).

$\Rightarrow$ a function <u>without</u> local maxima within D satisfies $\nabla^2 v > 0$ (v can have maxima on ∂D)

So let's construct $v = u + \epsilon f(x,y)$
where $\nabla^2 f$ = constant (positive) and $\epsilon > 0$.
and $f \geq 0$
(for example, $f(x,y) = x^2 + y^2$)

then
- $\max(v)$ is on the boundary
- $\max(u) = \max(v - \epsilon f(x,y))$

let $\epsilon \to 0$ so $\max(u)$ must be on the boundary too.

(3) <u>The mean value principle</u>

let D be a <u>planar</u> domain, let u be a harmonic function in D and (x_0, y_0) be a point within D.

Consider $R \in \mathbb{R}$ such that the disk D_R centred at (x_0, y_0) with radius R is fully contained in D. then

$$u(x_0, y_0) = \oint_{\partial D_R} u(x(s), y(s))\, d\ell \cdot \left[\oint_{\partial D_R} d\ell\right]^{-1}$$

= the average of u over the circle bounding D_R

<u>Note</u> here the parametrization of the circle is

$$x(s) = x_0 + R\cos(s) \rightarrow dx = -R\sin(s)ds$$
$$y(s) = y_0 + R\sin(s) \rightarrow dy = R\cos(s)ds$$

so $d\ell = \sqrt{dx^2 + dy^2} = R\,ds$ with $s \in [0, 2\pi]$ $\Rightarrow \left[\oint_{\partial D_R} d\ell\right]^{-1} = \frac{1}{2\pi R}$.

<u>Idea of Proof</u> Consider the function $\oint_{\partial D_r} u(x(s), y(s))\, d\ell \cdot \left[\oint_{\partial D_r} d\ell\right]^{-1} = F(r)$

then $F(r) = \frac{1}{2\pi \cancel{r}} \int_0^{2\pi} u(x_0 + r\cos(s), y_0 + r\sin(s))\, \cancel{r}\, ds$

$$\rightarrow \frac{\partial F}{\partial r} = \frac{1}{2\pi} \int_0^{2\pi} \underbrace{\frac{\partial}{\partial r}\left(u(x_0 + r\cos(s), y_0 + r\sin(s))\right) ds}_{\oint \partial_n u \quad d\ell}$$

$= 0$ since u is a harmonic function

$\Rightarrow F = \text{constant} = F(0) = u(x_0, y_0)$. so $F(R) = u(x_0, y_0)$

(4) Theorem (Strong maximum principle)

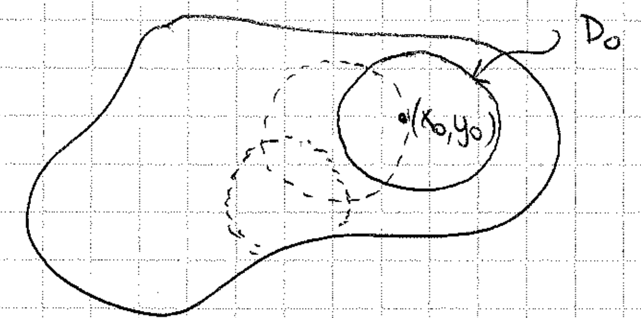
Let u be a harmonic function in a domain D. If u attains its maximum (or minimum) in D then u is constant

Idea behind the proof

If u attains its maximum somewhere at (x_0, y_0) within D, then construct a disk D_0 around (x_0, y_0) contained in D

By the mean value theorem & the fact that (x_0, y_0) is a maximum, we deduce that u is equal to the max everywhere on the contour of D_0. Since the MVT is also true for all disks within D_0, we conclude that $u = u_{max}$ for all points in D_0.

Finish the proof by "paving" D with connected disks.



7.4 Consequence: uniqueness of solutions in bounded domains for specific boundary conditions

Example for the Dirichlet problem

Consider the problem

$$\nabla^2 u = f(x,y) \quad \text{for } (x,y) \in D$$
$$u(x,y) = g(x,y) \quad \text{for } (x,y) \in \partial D$$

(where D is a bounded domain).

To prove uniqueness, consider two solutions v_1 and v_2 to the problem. Then

$v = v_1 - v_2$ is solution to $\begin{cases} \nabla^2 v = 0 & \text{in } D \\ v(x,y) = 0 & \text{on } \partial D \end{cases}$

Since v attains both minimum and maximum on ∂D then

$$0 \le v(x,y) \le 0 \quad \forall (x,y) \in D$$

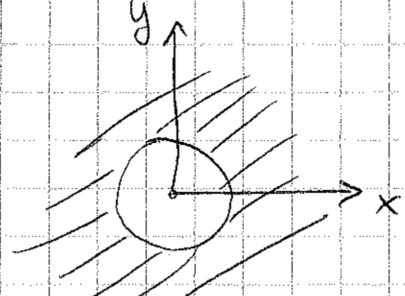
$\rightarrow$ v is identically 0 so $v_1 = v_2$

<u>Note</u>: Since the weak maximum principle only holds for bounded domains, the uniqueness of solutions to the Dirichlet problem only holds for bounded domains

<u>Counter example</u>

Consider $\nabla^2 u = 0 \quad : x^2+y^2 \ge 4$

$u(x,y) = 1 \quad x^2+y^2 = 4$



then $u(x,y) = 1$ is a solution

$$u(x,y) = \frac{\ln(x^2+y^2)}{2\ln 2}$$

is also a solution.

1A.5 Green's identities

Consider the divergence theorem:

$$\int_D \nabla \cdot \underset{\sim}{F}\, dV = \oint_{\partial D} \underset{\sim}{F} \cdot \underset{\sim}{n}\, ds$$

If $F = \nabla u$ then we get Green's first identity:

$$\int_D \nabla^2 u \, dV = \oint \underset{\sim}{\nabla u} \cdot \underset{\sim}{n} \, dS$$

If $F = v\nabla u - u\nabla v$ then we get Green's second identity (*)

$$\left.\begin{aligned} \int_D \nabla\cdot(v\nabla u - u\nabla v)\, dV &= \int_D (v\nabla^2 u - u\nabla^2 v)\, dV \\ &= \oint_{\partial D} (v\,\underset{\sim}{\nabla u}\cdot\underset{\sim}{n} - u\nabla v\cdot n)\, dS . \end{aligned}\right\} (*)$$

finally, we can integrate by parts

$$\begin{aligned} \int_D v\nabla^2 u \, dV &= \int_D v\, \nabla\cdot(\nabla u)\, dV \\ &= \int_D \nabla\cdot(v\nabla u) - \nabla u\cdot\nabla v \, dV \\ &= \int_{\partial D} v\,\underset{\sim}{\nabla u}\cdot\underset{\sim}{n}\, dS - \int_D \nabla u\cdot\nabla v \, dV \end{aligned}$$

so that

$$\int_D \underset{\sim}{\nabla u}\cdot\underset{\sim}{\nabla v}\, dV = \int_{\partial D} v\,\underset{\sim}{n}\cdot\nabla u\, dS - \int_D v\nabla^2 u\, dV .$$

$\Rightarrow$ the third Green's identity.

6.1.6 Application of Green's identity to the "uniqueness" of Neumann problems

Consider the problem
$$\nabla^2 u = f(x,y) \quad (x,y)\in D,$$
$$\hat{n}\cdot\nabla u = g(x,y) \quad (x,y)\in\partial D$$

then given two solutions v_1 and v_2 to the problem, construct
$$v = v_1 - v_2 .$$

Then v solves
$$\begin{cases} \nabla^2 v = 0 & (x,y)\in D \\ \hat{n}\cdot\nabla v = 0 & (x,y)\in\partial D . \end{cases}$$

Use Green's third identity with $u=v$ then
$$\int_V v\nabla^2 v \, dV = \int_{\partial V} v\hat{n}\cdot\nabla v \, dS - \int_V |\nabla v|^2 dV$$

$$\Rightarrow \int_V |\nabla v|^2 dV = 0 \rightarrow \nabla v = 0 \text{ everywhere}$$
$$\Rightarrow v \text{ is } \underline{\text{constant}}.$$

So if v_1 is a solution then any other function
$$v = v_1 + K \text{ is also a solution}.$$

Exercise: What happens in the case of Robin conditions?
$$\nabla^2 u = f(x,y) \quad (x,y)\in D.$$
$$u + \alpha\hat{n}\cdot\nabla u = g(x,y) \quad (x,y)\in\partial D$$

II Green's functions revisited

6.2.1 Fundamental solution of Laplace equation and applications

We saw that the function

$$\Gamma(x,y) = -\frac{1}{4\pi}\ln(x^2+y^2) = -\frac{1}{2\pi}\ln\sqrt{x^2+y^2}$$

is a solution of $\nabla^2\Gamma = 0$ everywhere in the plane (with no bcs) except at (x,y) where it is undefined.

<u>Definition</u>: $\Gamma(x,y;\xi,\eta) = -\frac{1}{4\pi}\ln\left((x-\xi)^2+(y-\eta)^2\right)$

is the fundamental solution of Laplace equation with a <u>pole</u> at (ξ,η)

6.2.2 Properties of Γ

Consider a domain D, and a function u solution of $\nabla^2 u = f$ in D, f smooth.

Then for all (ξ,η) in D then

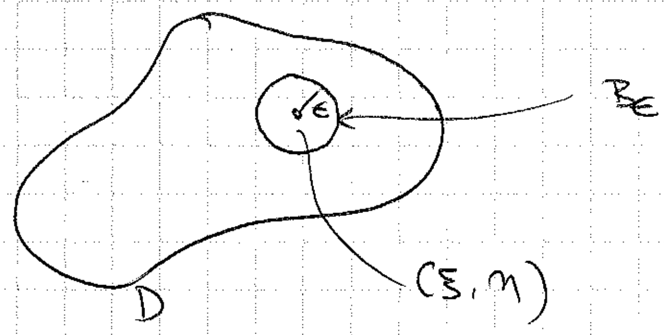
$$u(\xi,\eta) = \int_{\partial D}\left[\Gamma(x,y;\xi,\eta)\,\partial_n u - u\,\partial_n\Gamma(x,y;\xi,\eta)\right]dS$$

$$-\int_D \Gamma(x,y;\xi,\eta)f(x,y)\,dV$$

<u>Proof</u>: Use Green's #2 identity with u and Γ in the following domain:

$$D_\epsilon = D - B_\epsilon$$

where B_ϵ is a "sphere" of radius ϵ centred on (ξ, η)



then

$$\int_{D_\epsilon} \left(\Gamma \nabla^2 u - u \nabla^2 \Gamma\right) dV = \int_{\partial D_\epsilon} \left(\Gamma \partial_n u - u \partial_n \Gamma\right) dS.$$

in D_ϵ, $\nabla^2 \Gamma = 0$ everywhere.

$\partial D_\epsilon = \partial D - \partial B_\epsilon$ so

$$\int_{\partial D_\epsilon} \Gamma \partial_n u - u \partial_n \Gamma \, dS = \int_{\partial D} \left(\Gamma \partial_n u - u \partial_n \Gamma\right) dS - \int_{\partial B_\epsilon} \left(\Gamma \partial_n u - u \partial_n \Gamma\right) dS.$$

$$\int_{\partial B_\epsilon} \Gamma \partial_n u \, dS = \int_{\partial B_\epsilon} -\frac{1}{2\pi} \ln \epsilon \, \partial_n u \, dS = -\frac{1}{2\pi} \ln \epsilon \int_{\partial B_\epsilon} \partial_n u \, dS$$

$$= -\frac{1}{2\pi} \ln \epsilon \int_{B_\epsilon} \nabla^2 u \, dV$$

$$= -\frac{1}{2\pi} \ln \epsilon \int_{B_\epsilon} f \, dV = O\left(\epsilon^d \ln \epsilon\right)$$

where d is the spatial dimension

so $\displaystyle\int_{\partial B_\epsilon} \Gamma \partial_n u \, dS \to 0$ when $\epsilon \to 0$.

$$\int_{\partial B_\epsilon} u \partial_n \Gamma \, dS = \int_{\partial B_\epsilon} -\frac{u}{2\pi\epsilon} \, dS = -\int_0^{2\pi} \frac{u(\xi + \epsilon\cos\theta, \eta + \epsilon\sin\theta)}{2\pi\epsilon} \, \epsilon \, d\theta$$

since $\Gamma = -\frac{1}{2\pi} \ln(r - r_0)$

So $\displaystyle\int_{\partial B_\epsilon} u\,\partial_n \Gamma\, dS = -u(\xi,\eta)$ as $\epsilon \to 0$

so finally, taking the limit as $\epsilon \to 0$ of all terms we get

$$\int_D \Gamma \nabla^2 u \, dV = \int_{\partial D} \Gamma \partial_n u - u\,\partial_n \Gamma \, dS - u(\xi,\eta)$$

$\square$.

<u>Corollary</u>: For any domain D, $\displaystyle\int_{\partial D} \partial_n \Gamma \, dS = -1$ (use $u=1$).

6.2.3 Unbounded domains

Let's first consider the case of an un-bounded domain, with the required condition that
$u(x,y) \to 0$ as $|x^2+y^2| \to +\infty$.

In that case, the surface term vanishes and we have

$$u(\xi,\eta) = -\int_{\mathbb{R}^2} \Gamma \nabla^2 u \, dV = -\int_{\mathbb{R}^2} \nabla^2 \Gamma \; u(x,y)\,dx\,dy$$

$\Rightarrow$ But Γ is such that $\nabla^2 \Gamma = 0$ everywhere except at (ξ,η)

$\Rightarrow \nabla^2 \Gamma$ is a δ- function

$\Rightarrow$ more precisely $\boxed{\nabla^2 \Gamma = -\delta(x-\xi, y-\eta)}$

<u>Conclusions</u>:
Γ is the solution of $\nabla^2 \Gamma = -\delta(x-\xi, y-\eta)$

$u(\xi,\eta) = -\displaystyle\int_{\mathbb{R}^2} \Gamma(x,y;\xi,\eta)\, f(x,y)\,dx\,dy$ is the solution of $\nabla^2 u = f$.

Γ is (-) the Greens function in the plane

Examples of applications

- The gravitational potential created by a distribution of mass $\rho(\underline{r})$ is found by solving the Poisson equation
$$\nabla^2 \phi = 4\pi G \rho(\underline{r})$$

- In an unbounded domain, we require that $|\phi(\underline{r})| \to 0$ as $|\underline{r}| \to \infty$.

$\Rightarrow$ we can apply the previously derived formula!

① In a 2D plane, we know that
$$\Gamma(x,y) = -\frac{1}{4\pi} \ln(x^2+y^2)$$

so
$$\phi(x,y) = -\int_{\mathbb{R}^2} dx'dy' \, 4\pi G \rho(x',y') \Gamma(x-x', y-y')$$
$$= + \int_{\mathbb{R}^2} dx'\, dy' \, G \rho(x',y') \ln\left((x-x')^2 + (y-y')^2\right)$$

This may not look familiar, but it is the formula for a 2D gravitational potential!

② This is generalizable to higher-dimensions: eg: 3D

Idea: Γ is the center-symmetric solution of the Laplacian operator: in 3D, seek $\Gamma = \Gamma(r)$ only

$$\Rightarrow \quad \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Gamma}{\partial r}\right) = 0$$
$$\Rightarrow \quad r^2 \frac{\partial \Gamma}{\partial r} = K \quad \Rightarrow \quad \Gamma = -\frac{K}{r}$$

Again, we select a normalization such that

$$\Gamma(r) = +\frac{1}{4\pi r}$$

to guarantee, as in previous section, $\int_{\partial B_\epsilon} u \partial_n \Gamma \, dS = -u(\xi, \eta)$ (see above).

In cartesian coordinates: $\Gamma(x,y,z) = \dfrac{1}{4\pi\sqrt{x^2+y^2+z^2}}$

$\Rightarrow$ Now we know that in 3D

$$\phi(x,y,z) = -\int_{\mathbb{R}^3} \frac{dx'\,dy'\,dz' \; G\rho(x',y',z')}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$$\phi(\underline{r}) = -\int \frac{d^3\underline{r}' \; G\rho(\underline{r}')}{|r-r'|}$$

↑ the standard formula for the 3D gravitational potential ϕ generated by a distribution of mass $\rho(\underline{r})$.