Review Dirac Delta function $\delta(x)$

① Definition. The Dirac Delta function $\delta(x)$ is defined as having the following properties

① $$\int_{-\infty}^{+\infty} \delta(x)dx = 1$$

(Note that ② implies ①)

② $$\int_{-\infty}^{+\infty} \delta(x) f(x) dx = f(0) \quad \text{for any function } f(x)$$

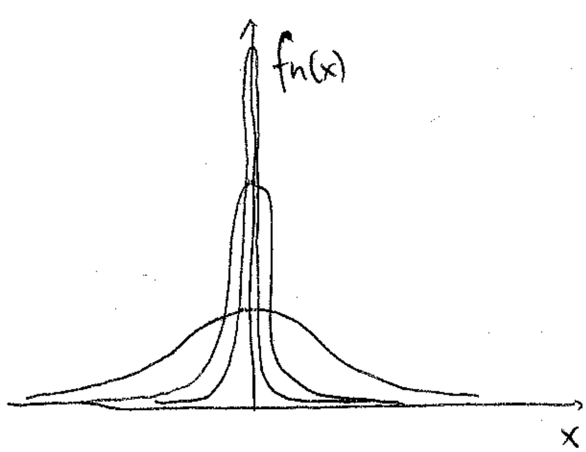
③ $$\delta(x) = 0 \quad \forall x \neq 0.$$

In fact, $\delta(x)$ is not properly speaking a function

It can be considered as the limit of the sequence of functions

$$f_n(x) = \frac{n}{\sqrt{2\pi}} e^{-\frac{n^2x^2}{2}} \qquad n \geq 1, \; n \to \infty.$$

for example



Note: It can also be considered as a limit of many other functions:

$$\lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + x^2}$$

$$\lim_{\epsilon \to 0} \frac{1}{\pi x} \sin\left(\frac{x}{\epsilon}\right)$$

② Fourier transform

The Fourier transform of the Dirac Delta function is

$$D(\omega) = \int_{-\infty}^{+\infty} e^{i\omega x} \delta(x) dx = 1$$

$\to$ the FT of the Dirac Delta is constant.

By inverse transform we also see that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} D(\omega)e^{-i\omega x}\,d\omega = \delta(x)$$

$$\Rightarrow \quad \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega x}\,d\omega = \delta(x)$$

Note that if we define

$$f_n(x) = \frac{1}{2\pi}\int_{-n}^{n} e^{-i\omega x}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-n}^{n} \cos\omega x + i\sin\omega x \; d\omega$$

$$= \frac{1}{2\pi}\left[\frac{1}{\omega}\sin\omega x - \frac{i}{\omega}\cos\omega x\right]_{-n}^{n}$$

$$= \frac{1}{2\pi}\left[\frac{2}{n}\sin nx\right] = \frac{1}{n\pi}\sin nx$$

$$= \frac{\epsilon}{\pi}\sin\left(\frac{x}{\epsilon}\right) \quad \text{if } \epsilon = \frac{1}{n}$$

→ We recover the idea of

$$\delta(x) = \lim_{n\to\infty} f_n(x) = \lim_{\epsilon\to 0}\frac{\epsilon}{\pi}\sin\left(\frac{x}{\epsilon}\right).$$

CHAPTER 6 Elliptic equations & Green's Functions

I Green's functions in ODES

Consider the linear, forced ODE

$$a(x)\,u_{xx} + b(x)\,u_x + c(x)\,u = f(x)$$

with some homogeneous initial or boundary conditions:
- either initial conditions (at $x = x_0$)
- or 2pt boundary conditions ($x=a$ and $x=b$)

Define the Greens function $G(x,x')$ as the solution of

$$a(x)G_{xx} + b(x)G_x + c(x)G = \delta(x-x')$$

under the same set of conditions.

Then (a) $u(x) = \int_{x_0}^{\infty} f(x')\,G(x,x')\,dx'$ for an IVP.

or

(b) $u(x) = \int_a^b f(x')\,G(x,x')\,dx'$ for a 2-pt boundary-value problem

Proof: integral bounds either $(x_0, +\infty)$ or (a,b)

$$a(x)\left[\int f(x')G(x,x')dx'\right]_{xx} + b(x)\left[\int f(x')G(x,x')dx'\right]_x$$

$$+ c(x)\int f(x')G(x,x')dx' =$$

$$\int f(x')\left[aG_{xx} + bG_x + cG\right]dx'$$

$$= \int f(x')\,\delta(x-x')\,dx' = f(x)$$

$\Rightarrow$ u satisfies the ODE.

Boundary conditions:

Suppose we had required $u(a)=0$, G must satisfy $G(a,x')=0$

$\Rightarrow$

$$u(a) = \int G(a,x')\, f(x')\, dx' = 0 \quad \text{as required.}$$

Suppose we had required $u'(a)=0 \Rightarrow G$ must satisfy $\left.\frac{dG}{dx}\right|_{x=a} = 0$

$$\left.\frac{du}{dx}\right|_{x=a} = \int \frac{dG}{dx}(a,x')\, f(x')\, dx' = 0 \quad \text{as required.}$$

Note: See textbook for non-homogeneous bc case; the trick is to change variable u to another function which has homogeneous bcs.

Examples.

(A). Impulse response

Consider the harmonic oscillator

$$u_{tt} + \omega^2 u = f(t) \quad , \quad u(0) = u_t(0) = 0$$

$\to$ Let's solve it for an impulse $\delta(t-t')$ function

$$G_{tt} + \omega^2 G = \delta(t-t')$$

Laplace transform $\Rightarrow$

$$\omega^2 \hat{G} + s^2 \hat{G}(s) - sG(0) - G'(0) = \int_0^\infty e^{-st}\, \delta(t-t')\, ds = e^{-st'}$$

$$\Rightarrow \quad \hat{G}(s) = \frac{e^{-st'}}{s^2+\omega^2} \qquad \text{since } G(0) = G'(0) = 0$$

So $G(t,t') = \begin{cases} \dfrac{\sin(\omega(t-t'))}{\omega} & t>t' \\ 0 & t<t' \end{cases}$

So $u(t) = \displaystyle\int_0^{\infty} f(t')\, G(t,t')\, dt'$

$= \displaystyle\int_0^{\infty} f(t') \frac{\sin(\omega(t-t'))}{\omega}\, dt'.$

$G(t,t')$ can be viewed as the solution to the ODE forced with an impulse (δ-function forcing)

The solution to continuous forcing then becomes a linear superposition of the responses to the impulses

For example, if $f(t) = 1$ for $t \in [1,2]$ then

$$u(t) = \int_1^2 \frac{\sin(\omega(t-t'))}{\omega} H(t-t')\, dt'$$

↳ if $t<1$ $u(t)=0$

↳ if $t\in[1,2]$ then

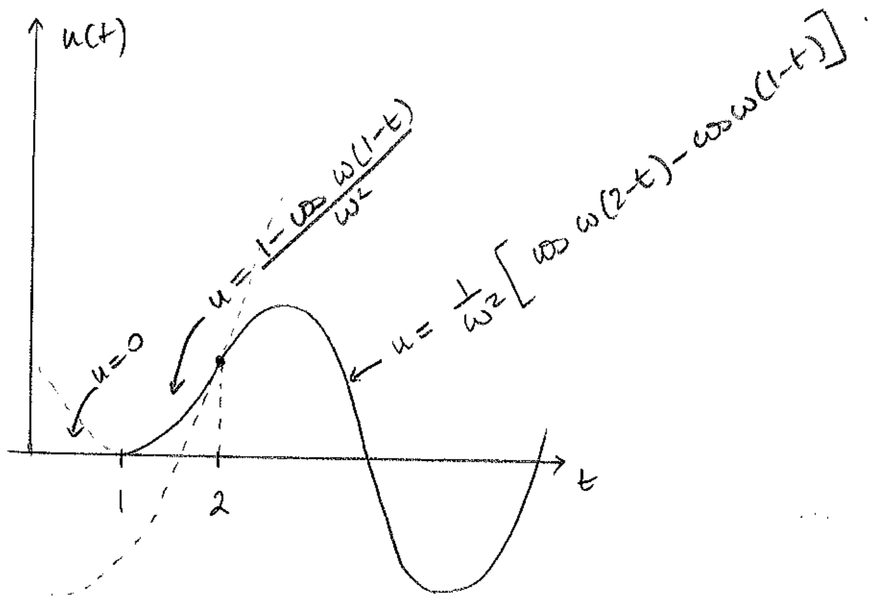
$$u(t) = \int_1^t \frac{\sin(\omega(t-t'))}{\omega}\, dt'$$

$$= \int_0^{1-t} \frac{\sin \omega x}{\omega}\, dx = \frac{1}{\omega^2}\left[1-\cos\omega(1-t)\right]$$

↳ if $t>2$ then

$$u(t) = \int_1^2 \frac{\sin(\omega(t-t'))}{\omega}\, dt'$$

$$= \frac{1}{\omega^2}\left[\cos\omega(2-t) - \cos\omega(1-t)\right].$$



(B) <u>2-pt BVP</u>

Let's now look at the following problem

$$\begin{cases} u_{xx} + u = f(x) \\ u(0) = u(\frac{\pi}{2}) = 0 \end{cases} \qquad \text{(choose } L = \tfrac{\pi}{2} \text{ for simplicity)}$$

- If we solve this using eigenmodes expansions we get

$$v_n(x) = \sin(2n\,x)$$

$$\lambda_n = 4n^2 - 1$$

and $$G(x,x') = \sum_{n=0}^{\infty}{}' -\frac{4}{\pi} \frac{\sin(2nx)\sin(2nx')}{4n^2-1}$$

(see previous lecture)

and $$u(x) = \int_0^L f(x')G(x,x')\,dx'$$

Is there another way of achieving the same result?

$\Rightarrow$ yes.

- Suppose we want to solve for

$$G_{xx} + G = \delta(x-x')$$

Unless $x = x'$, $G_{xx} + G = 0 \Rightarrow$ the solutions are linear combinations of $\sin x$ and $\cos x$.

let's construct a function which is piecewise differentiable, satisfies $G(0) = G(\frac{\pi}{2}) = 0$, and is only non-differentiable at $x = x'$.

$\Rightarrow$ On the $[0, x']$ interval $G(x) = A\sin x$ to satisfy $G(0) = 0$

on $[x', \frac{\pi}{2}]$ interval then $G(x) = B\cos x$ (to satisfy $G(\frac{\pi}{2}) = 0$)

(Note that if $b = L$ instead of $\frac{\pi}{2}$ the method is the same but the algebra trickier – HW!)

Now, let's require continuity of this function at $x = x'$ $\Rightarrow$

$$A\sin(x') = B\cos(x')$$

$$\Rightarrow \quad A = \frac{B}{\tan x'}$$

Finally, to get the amplitude we note that

$$\int_0^{\frac{\pi}{2}} dx \left[G_{xx} + G\right] = \int_0^{\frac{\pi}{2}} \delta(x-x')\,dx = 1$$

$$\Rightarrow \quad 1 = \left[G_x\right]_0^{\frac{\pi}{2}} + \int_0^{x'} A\sin x\,dx + \int_{x'}^{\frac{\pi}{2}} B\cos x\,dx$$

$$1 = -B\sin\left(\tfrac{\pi}{2}\right) - A\cos(0) + \left[-A\cos x\right]_0^{x'} + \left[B\sin x\right]_{x'}^{\frac{\pi}{2}}$$

$$1 = -A\cos x' - B\sin x'$$

These two equations yield A & B:

$$\begin{cases} A = -\cos x' \\ B = -\sin x' \end{cases}$$

So finally

$$G(x,x') = \begin{cases} -\cos x' \sin x & \text{for } x \in [0, x') \\ -\sin x' \cos x & \text{for } x \in [x', \frac{\pi}{2}] \end{cases}$$

The two functions $G(x,x')$ are the same (see Maple).

$\Rightarrow$ Properties of Greens functions: note that they do not have to be differentiable everywhere.

Generally: For a n-th order ODE, the Green's function is differentiable $n-2$ times

Bottom Line: There are $\neq$ ways of constructing the Green's functions. In what follows, we will see some new ways in 2D, with specific application to the Poisson equation

$$\nabla^2 u = f(x,y).$$

We will also explore in the meantime some other properties of elliptic equations