(4) Fundamental 1^{st} order PDE: the transport equation

$$\boxed{\frac{\partial f}{\partial t} + \nabla \cdot (\underline{u} f) = 0}$$

where $\underline{u}$ is a velocity field (known)

Expression in Cartesian coordinates:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(u f) + \frac{\partial}{\partial y}(v f) + \frac{\partial}{\partial z}(w f) = 0$$

if

$$\underline{u} = (u, v, w)$$

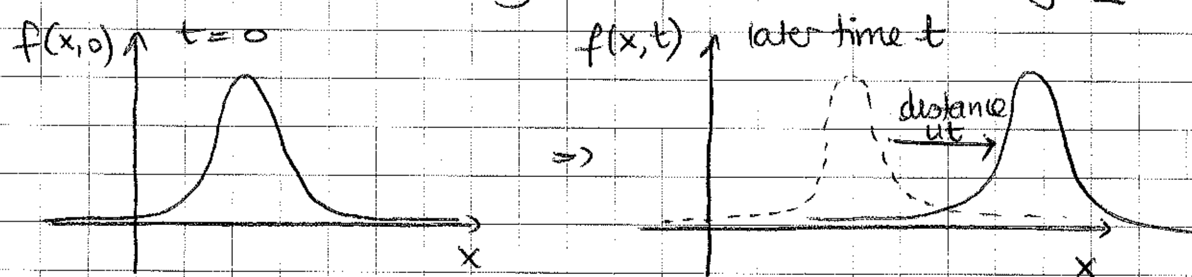
- This equation expresses the conservation of the quantity f through transport by the velocity field $\underline{u}$.

- If $\underline{u}$ is a constant field then

$$\boxed{\frac{\partial f}{\partial t} + \underline{u} \cdot \nabla f = 0}$$

(note that this is also generally true for all vector fields $\underline{u}$ s.t. $\nabla \cdot u = 0$)

A solution is merely "moved around" by $\underline{u}$



(note the similarity with propagation of wave)

See ppt for movies in more than 1D

(5) Additional conditions & well-posedness (a first look at)

- When modelling physical problems, the PDE is <u>always</u> accompanied by additional conditions, usually in the form of
  - initial conditions (for a time-dependent problem)
  - boundary conditions (for a problem on a finite domain)
  - regularity conditions (either, regularity/bound at infinity, or regularity at a coordinate singularity)

  The behaviour of a solution depends as much of the PDE than on these additional conditions

- For a given PDE with given additional conditions, there can be no, one or many possible solutions

  <u>example</u>. $u_t = u_x$

  - Given this PDE <u>without</u> any additional conditions, there are an infinity of solutions ($u = c$ for all $c \in \mathbb{R}$)
  - Given this PDE together with
    $$u(x, t=0) = \phi(x)$$
    there is a unique solution (see Chapter 2).
  - Given this PDE with $u(x, t=x) = \phi(x)$ then there is no solution (see Chapter 2) or an infinite # of solutions.

- For a given equation and set of additional conditions, a small change in the parameters of the equation or of the conditions can lead to a <u>big</u> change in the solution.

Example $u_t = -u_{xx}$ $t>0$
$u(x,0) = 1$

→ obvious solution is $u(x,t) = 1$
but if we had chosen $u(x,0) = 1 + \frac{1}{n} \cdot \sin(nx)$
then the solution is

$$u(x,t) = 1 + \frac{1}{n} e^{n^2 t} \sin(nx) \qquad \text{(CHECK THIS)}$$

Now for n large enough $1 + \frac{1}{n}\sin(nx) \simeq 1$
but after a time t large enough $\frac{e^{n^2 t}}{n} \gg 1$
so a small difference in initial conditions creates an enormous difference in the final solution.

Definition:
a PDE (or set of PDE) and its associated additional conditions is a well-posed problem if
- it has a solution
- this solution is unique
- the structure of the solution is unchanged by infinitesimal variations of the parameters and/or of the additional conditions.

Typically : • a well-thought, well-modelled physical problem will result in a well-posed problem. because in nature, the solution exists.
(although simplifying assumptions & shortcuts often lead to ill-posed problems)

BUT : not necessarily always the case. For nonlinear PDEs :
- multiple solutions can exist in which slight changes in the boundary conditions lead to one, or the other equilibrium
- sometimes, it is the discontinuous solutions that interest us (@shock physics)

⇒ called weak solutions

CHAPTER 2 First order PDEs (in 2 dimensions)

2.1 General formulae

A first order PDE in 2 dimensions is in the form of

$$F\left(x, t, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) = 0$$

- A first order <u>linear</u> PDE in 2 dimensions is

$$a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t)u + d(x,t).$$

NONLINEAR PDEs:

- A first order <u>semilinear</u> PDE in 2D is

$$a(x,t)\frac{\partial u}{\partial t} + b(x,t)\frac{\partial u}{\partial x} = c(x,t,u)$$

- A first order quasilinear PDE is

$$a(x,t,u)\frac{\partial u}{\partial t} + b(x,t,u)\frac{\partial u}{\partial x} = c(x,t,u).$$

A fully nonlinear first order PDE is none of the above!

2.2 Method of characteristics for quasilinear equations

2.2.1 Warmup example

Let's study $u_t = c_0 u + G(x,t)$ <u>c_0 constant</u>

Note that for each x, it is actually an ODE in t
→ fix x and solve it!

Use integrating factor method (for example)

$$u_t - c_0 u = G(x,t)$$

⇒ We try to find an integrating factor $\mu(x,t)$ such that

$$\mu u_t - \mu c_0 u = \mu q(x,t)$$

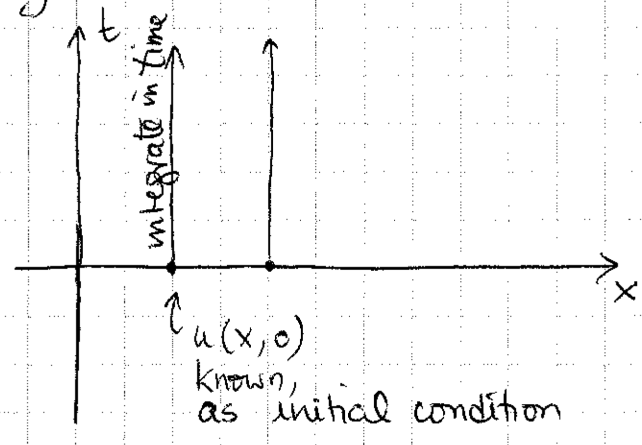
$$= \frac{\partial}{\partial t}(\mu u)$$

$\longrightarrow$ take $\mu = e^{-c_0 t}$ so

$$\frac{\partial}{\partial t}\left(e^{-c_0 t} u\right) = e^{-c_0 t} q(x,t)$$

$$e^{-c_0 t} u(x,t) - e^{-c_0 \cdot 0} u(x,0) = \int_{t'=0}^{t'=t} e^{-c_0 t'} q(x,t')\,dt'$$

Again, this can be done for each value of x separately: we are solving the equation by integrating along lines of constant x



<u>Initial conditions</u> (u is known at $t=0$)

Suppose we require that $u(x,0) = 3x$ then

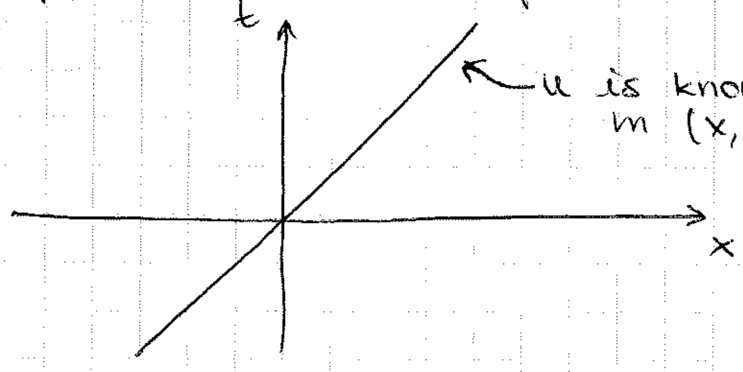
$$u(x,t) = e^{+c_0 t} u(x,0) + \int_{t'=0}^{t'=t} e^{-c_0(t'-t)} q(x,t')\,dt'$$

$$= 3x e^{+c_0 t} + \int_{t'=0}^{t'=t} e^{-c_0(t'-t)} q(x,t')\,dt'$$

$\longrightarrow$ a unique solution.

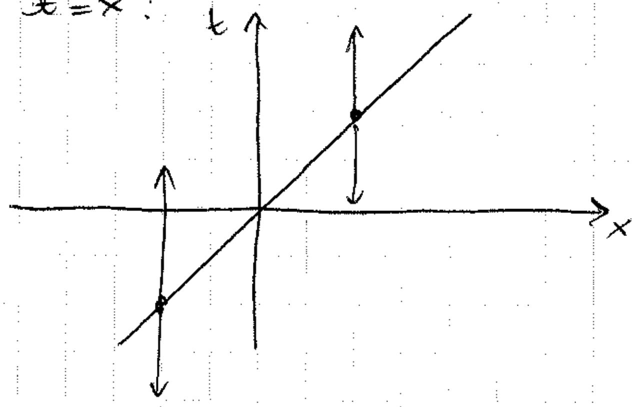
Other kinds of additional condition

① Suppose instead we require that $u(x,x) = 3x$



← u is known on this line in (x,t) plane

Then, instead of integrating from $t'=0$, we integrate from $t'=x$:



Mathematically:

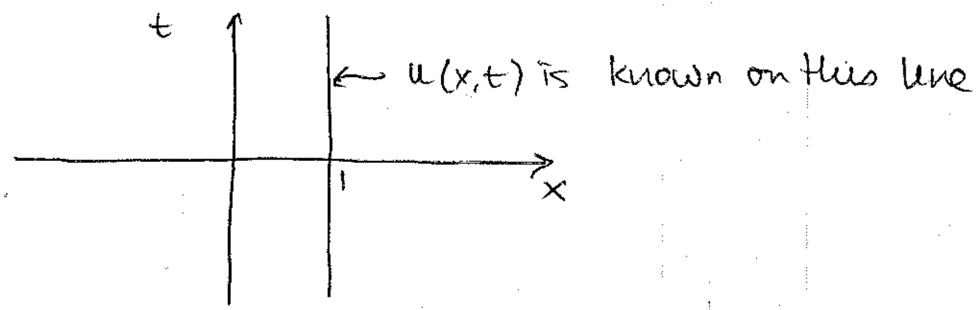
$$e^{-c_0 t}\, u(x,t) - e^{-c_0 x}\, u(x,x) = \int_{t'=x}^{t'=t} e^{-c_0 t'}\, G(x,t')\, dt'$$

$$\Rightarrow \quad e^{-c_0 t}\, u(x,t) = e^{-c_0 x} \cdot 3x + \int_x^t e^{-c_0 t'}\, G(x,t')\, dt'$$

$$u(x,t) = e^{-c_0 (x-t)} \cdot 3x + \int_x^t e^{-c_0 (t'-t)}\, G(x,t')\, dt'$$

→ again, there is a unique solution to the PDE with the given additional condition.

② Now suppose we set $G = 0$ and try to impose as additional condition $u(1,t) = 2t$



Problem! The additional condition doesn't satisfy the equation

$$\frac{\partial u}{\partial t} = 2 \quad \rightarrow \quad u_t - C_0 u = 2 - 2C_0 t \neq 0 .$$

$\rightarrow$ there are no solutions to the equation!

③ Now suppose $u(1,t) = 2e^{C_0 t}$ then

$$u_t - C_0 u = 2C_0 e^{C_0 t} - 2C_0 e^{C_0 t} = 0 \quad \checkmark$$

$\Rightarrow$ the additional condition satisfies the equation

But note that any function of the form
$$u(x,t) = f(x)\, e^{C_0 t}$$
satisfies the PDE <u>and</u> the additional condition provided $f(1) = 2$

$\Rightarrow$ there are an ∞ of solutions to the problem!

<u>Conclusion</u>: • Depending on the additional conditions chosen, there can be one, no or an ∞ of solutions to the problem. Case ① is <u>well-posed</u> while cases ② and ③ are <u>ill-posed</u>

• What is the difference between cases ①, ② and ③?

Note that in case ①, the additional condition crosses all lines of constant x, while in cases ② and ③, the additional condition <u>is</u> a line of constant x.

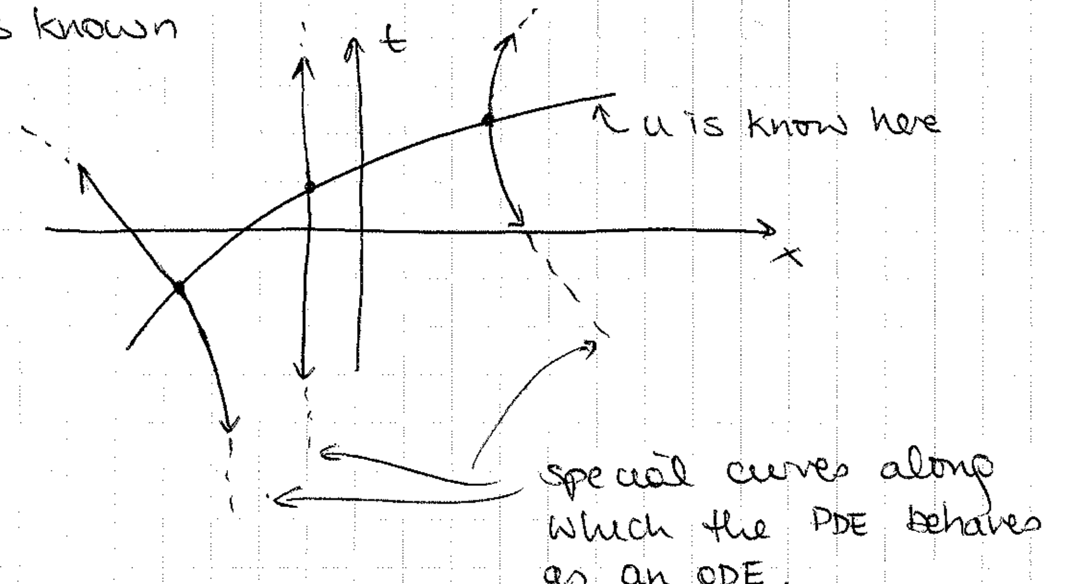
2.2.2 Gearing up to the general method

Now consider the linear transport equation with constant coefficients.

$$a \frac{\partial u}{\partial t} + b \frac{\partial u}{\partial x} = c_1 u + c_0$$

where a, b, c_1, c_0 are constants.

<u>Idea</u>: We would like to find curves (as before) along which we could integrate the PDE <u>as if</u> it were an ODE, from an initial or additional condition line where $u(x,t)$ is known



DETOUR: **<u>Review of parametric curves</u>**

Any curve in $\mathbb{R}^n$ can be represented by a set of parametric equations

$$\begin{cases} x_1 = f_1(s) \\ x_2 = f_2(s) \\ \vdots \\ x_n = f_n(s) \end{cases}$$

where s is the parameter.

<u>Examples:</u> A circle in $\mathbb{R}^2$ (x,y) centred on $(0,0)$ has the equation $\begin{cases} x = R\cos(s) \\ y = R\sin(s) \end{cases}$ where R is the radius

- A straight line in $\mathbb{R}^2$ has the parametric equation

$$\begin{cases} x = as + c \\ y = bs + d \end{cases}$$

<u>Check</u>: eliminate s to get

$$y = b\left(\frac{x-c}{a}\right) + d = \frac{b}{a}x + \left(d - \frac{bc}{a}\right)$$

<u>Property of parametric curves</u>

The tangent vector to the curve $\{f_1(s) \dots f_n(s)\}$ is

$$\underline{df} = \begin{pmatrix} df_1/ds \\ df_2/ds \\ \vdots \\ df_n/ds \end{pmatrix}$$

<u>Examples</u>: • the tangent vector to the line

$$\begin{cases} x = as + c \\ y = bs + d \end{cases} \quad \text{is} \quad \underline{df} = \begin{pmatrix} dx/ds \\ dy/ds \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

• Suppose you are travelling from SC to Big Sur. Your <u>trajectory</u> is given by the parametric curve

$$\begin{pmatrix} x(t) \\ y(t) \\ h(t) \end{pmatrix} \begin{matrix} \leftarrow \text{latitudinal position } x \\ \leftarrow \text{longitudinal position } y \\ \leftarrow \text{height} \end{matrix}$$

Your <u>velocity</u> is the tangent vector to the trajectory

$$\underline{v} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dh}{dt} \end{pmatrix} \begin{matrix} \leftarrow \text{North-South velocity} \\ \leftarrow \text{East-West velocity} \\ \leftarrow \text{vertical velocity.} \end{matrix}$$

<u>Note</u>: A parametrization is <u>NOT</u> unique.

<u>Example</u>: $\begin{cases} x = R\sin s \\ y = R\cos s \end{cases}$ and $\begin{cases} x = R\sin(s^2) \\ y = R\cos(s^2) \end{cases}$ represent the <u>same</u> curve