Hyperbolic case : similarly

$$\sum_n' \ddot{a}_n(t)\, v_n(x) + \sum' \lambda_n a_n(t)\, v_n(x) = \sum' b_n(t)\, v_n(x)$$

so by orthogonality

$$\ddot{a}_n(t) + \lambda_n a_n(t) = b_n(t)$$

This time we use the Laplace transform method:

$$s^2 \hat{a}_n - s a_n(0) - a_n'(0) + \lambda_n \hat{a}_n = \hat{b}_n(s)$$

$$\Rightarrow \quad \hat{a}_n(s) = \frac{\hat{b}_n(s) + s a_n(0) + a_n'(0)}{s^2 + \lambda_n}$$

Now, the λ_n are positive $\Rightarrow$ the inverse Laplace transform (see Maple) is

$$a_n(t) = \frac{1}{\sqrt{\lambda_n}} \int_0^t b_n(t') \sin\left(\sqrt{\lambda_n}\,(t-t')\right) dt'$$

$$+ \; a_n(0) \cos\left(\sqrt{\lambda_n}\, t\right) + a_n'(0) \frac{1}{\sqrt{\lambda_n}} \sin\left(\sqrt{\lambda_n}\, t\right)$$

$\Rightarrow$ The general solution of the problem becomes

$$u(x,t) = \sum_{n=0}^{\infty} v_n(x) \cdot \left[\frac{1}{\sqrt{\lambda_n}} \int_0^t b_n(t') \sin\left(\sqrt{\lambda_n}\,(t-t')\right) dt' \right.$$

$$\left. + \; a_n(0) \cos\left(\sqrt{\lambda_n}\, t\right) + a_n'(0) \frac{1}{\sqrt{\lambda_n}} \sin\left(\sqrt{\lambda_n}\, t\right) \right]$$

but with $b_n(t') = \displaystyle\int_a^b \frac{F(x',t')\, v_n(x')\, r(x')\, dx'}{\| v_n^2 \|}$

we get $$u(x,t) = \sum' \left[a_n(0) \cos\left(\sqrt{\lambda_n}\, t\right) + a_n'(0) \frac{1}{\sqrt{\lambda_n}} \sin\left(\sqrt{\lambda_n}\, t\right) \right] \cdot v_n(x)$$

$$+ \int_0^t \int_a^b F(x',t')\, G(x,x';t,t')\, dx'\, dt'$$

with

$$G(x,x';t,t') = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\lambda_n}} \sin\left(\sqrt{\lambda_n}(t-t')\right) \frac{v_n(x)\, v_n(x')\, r(x')}{\| v_n^2 \|}$$

↳ the wave kernel

Example of the bridge

Recall: $u_{tt} - c^2 u_{xx} = \sin\left(\frac{2\pi x}{L}\right)\cos(\omega t)$

$u = 0$ at both ends

$u_t(x,0) = u(x,0) = 0$

$\Rightarrow$ Eigenmodes/values of spatial homogeneous pb:

$$\begin{cases} v_n(x) = \sin\left(\frac{n\pi x}{L}\right) \\ \lambda_n = \frac{n^2\pi^2 c^2}{L^2} \end{cases}$$

then $u(x,t) = \sum \left[a_n(0)\cos\left(\frac{n\pi c t}{L}\right) + a_n'(0)\frac{L}{n\pi c}\sin\left(\frac{n\pi c t}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$

$$+ \int_0^t \int_0^L F(x',t')\, G(x,x';t,t')\, dx'\, dt'$$

Fitting this to ICs $\Rightarrow$ $a_n(0) = a_n'(0) = 0$

$$u(x,t) = \int_0^t dt' \int_0^L dx' \sin\left(\frac{2\pi x'}{L}\right)\cos(\omega t') \sum_{n=0}^{\infty} \frac{L}{n\pi c}\sin\left(\frac{n\pi c}{L}(t-t')\right) \frac{\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x'}{L}\right)}{\frac{L}{2}}$$

$$= \int_0^t dt' \cos(\omega t') \frac{L}{2\pi c}\sin\left(\frac{2\pi c}{L}(t-t')\right)\sin\left(\frac{2\pi x}{L}\right)$$

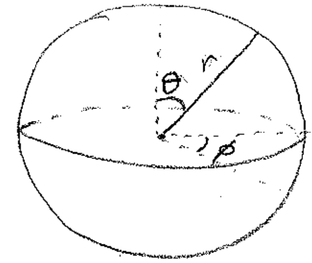
$$= \int_0^t \frac{dt'}{2}\left[\sin\left(\omega t' + \frac{2\pi c}{L}(t-t')\right) - \sin\left(\omega t' - \frac{2\pi c}{L}(t-t')\right)\right]\sin\frac{2\pi x}{L}\,\frac{L}{2\pi c}$$

$$= \frac{1/2}{\omega + \frac{2\pi c}{L}}\left[\cos\left(\frac{2\pi c t}{L}\right) - \cos\omega t\right] - \frac{1/2}{\omega - \frac{2\pi c}{L}}\left[\cos\left(\frac{2\pi c}{L}\right) - \cos\omega t\right]\frac{L}{2\pi c}$$

$$= \frac{1}{\omega^2 - 4\pi^2c^2/L^2}\left[\cos\left(\frac{2\pi c t}{L}\right) - \cos\omega t\right]\sin\left(\frac{2\pi x}{L}\right) \checkmark \qquad \cdot \sin\left(\frac{2\pi x}{L}\right)$$

Example of application to 3D problems

Solar oscillations



$0 \le r \le R_\odot$

- The Sun is a spherical ball of self-gravitating gas.
- The outer layer is convecting; the convective motions excite sound waves which propagate throughout the interior. The sound waves satisfy the approximate equation

$$\frac{\partial^2 p}{\partial t^2} = c_s^2(r)\,\nabla^2 p$$

p = pressure

c_s = sound speed, assumed to depend only on radius r

∇^2 = Laplacian in spherical coordinates r, θ, ϕ

$$\Rightarrow \frac{\partial^2 p}{\partial t^2} = c_s^2(r)\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial p}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 p}{\partial\phi^2}\right]$$

We will assume that the boundary conditions are

$$\begin{cases} p(0,\theta,\phi,t) < +\infty \\ p(R_\odot,\theta,\phi,t) = 0 \end{cases}$$

Eigenmodes

Separation of variables $\Rightarrow$ assume

$$p(r,\theta,\phi,t) = A(r)\,B(\theta)\,C(\phi)\,D(t)$$

$$\Rightarrow \underbrace{\frac{1}{D}\frac{d^2D}{dt^2}}_{\text{a function of time only}} = \underbrace{c_s^2(r)\left[\frac{1}{A}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dA}{dr}\right) + \frac{1}{B}\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dB}{d\theta}\right) + \frac{1}{C}\frac{1}{r^2\sin^2\theta}\frac{d^2C}{d\phi^2}\right]}_{\text{a function of space only}}$$

$$= -\omega^2$$

expect oscillations so select a negative constant

$$\Rightarrow \begin{cases} \dfrac{d^2 D}{dt^2} = -\omega^2 D & ① \\[2ex] \dfrac{1}{A}\dfrac{d}{dr}\left(r^2 \dfrac{dA}{dr}\right) + \dfrac{1}{B\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta \dfrac{dB}{d\theta}\right) + \dfrac{1}{C\sin^2\theta}\dfrac{d^2 C}{d\phi^2} = \dfrac{-\omega^2 r^2}{c_s^2(r)} & ② \end{cases}$$

take ②

$$\Rightarrow \underbrace{\frac{1}{A}\frac{d}{dr}\left(r^2 \frac{dA}{dr}\right) + \frac{\omega^2 r^2}{c_s^2(r)}}_{\text{A function of } r \text{ only}} = \underbrace{-\left[\frac{1}{B\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dB}{d\theta}\right) + \frac{1}{C\sin^2\theta}\frac{d^2 C}{d\phi^2}\right]}_{\text{A function of } \theta \text{ and } \phi \text{ only}}$$

$$= -\lambda$$

another constant

so
$$\begin{cases} \dfrac{d}{dr}\left(r^2 \dfrac{dA}{dr}\right) + \lambda A = -\dfrac{\omega^2 r^2}{c_s^2(r)} A & ③ \\[2ex] \dfrac{1}{B\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta \dfrac{dB}{d\theta}\right) + \dfrac{1}{C\sin^2\theta}\dfrac{d^2 C}{d\phi^2} = \lambda & ④ \end{cases}$$

take ④, multiply by $\sin^2\theta$ $\rightarrow$

$$\underbrace{-\lambda\sin^2\theta + \frac{1}{B}\sin\theta\frac{d}{d\theta}\left(\sin\theta \frac{dB}{d\theta}\right)}_{\text{A function of } \theta \text{ only}} = \underbrace{-\frac{1}{C}\frac{d^2 C}{d\phi^2}}_{\text{A function of } \phi \text{ only}}$$

$$= m^2$$

a constant.

Since we expect 2π-periodic functions in ϕ, we can straightaway recognize that constant to be m^2, with m integer.

So
$$\begin{cases} \dfrac{d^2C}{d\phi^2} = -m^2 C & (5) \\[2ex] \sin\theta \dfrac{d}{d\theta}\left(\sin\theta \dfrac{dB}{d\theta}\right) - m^2 B = \lambda \sin^2\theta\, B & (6) \end{cases}$$

$\Rightarrow$ (1), (3), (5) and (6) are the 4 equations representing the eigenmodes in the t, r, θ and ϕ variables.

<u>Solution in ϕ</u>: naturally $C_m(\phi) = \left\{ \begin{matrix} \cos m\phi \\ \sin m\phi \end{matrix} \right\}$

<u>Solution in θ</u>: The θ-mode depends on the value of m

<u>Trick</u> Let $\cos\theta = \mu$

then
$$\frac{d}{d\theta} = \frac{d}{d\mu}\frac{d\mu}{d\theta} = -\sin\theta \frac{d}{d\mu}$$

so (6) becomes

$$(1-\mu^2)\frac{d}{d\mu}\left((1-\mu^2)\frac{dB}{d\mu}\right) - m^2 B = \lambda(1-\mu^2) B$$

$$\longrightarrow \quad \frac{d}{d\mu}\left((1-\mu^2)\frac{dB}{d\mu}\right) - \frac{m^2}{1-\mu^2} B = \lambda B$$

The solutions to this equation with $\lambda = -\ell(\ell+1)$ are the Legendre functions
$$\left\{ P_\ell^m(\mu) \right\}_{\ell, m \text{ integers}}$$

So
$$B_\ell^m(\theta) = \left\{ P_\ell^m(\cos\theta) \right\}_{m, \ell \text{ integers}}$$

Some properties of Legendre functions:

- $$P_\ell^m(x) = (-1)^{|m|}(1-x^2)^{\frac{|m|}{2}} \frac{d^{|m|} P_\ell(x)}{dx^{|m|}}$$
  where $P_\ell(x)$ is the Legendre Polynomial of order ℓ

- $P_\ell(x) = P_\ell^0(x)$
- $P_0(x) = 1$

  $P_1(x) = x$

$$(n+1) \quad P_{n+1}(x) = (2n+1)x P_n(x) - n P_{n-1}(x)$$

- So $P_\ell^m(x) = 0 \quad \forall\ |m| > \ell$

<u>Note</u>: The combination of the two angular functions are usually called <u>spherical harmonics</u> and noted $Y_\ell^m(\theta,\phi)$

e.g. $Y_2^1(\theta,\phi) \propto P_2^1(\cos\theta)\cos\phi$

although standard conventions usually use

$$Y_2^1(\theta,\phi) \propto P_2^1(\cos\theta)e^{i\phi}$$

$$Y_\ell^m(\theta,\phi) \propto P_\ell^m(\cos\theta)e^{im\phi}$$

<u>Solution in r</u>

$$\frac{d}{dr}\left(r^2\frac{dA}{dr}\right) - \ell(\ell+1)A = -\frac{\omega^2 r^2}{c_s^2(r)}A$$

$\rightarrow$ clearly, this equation cannot be solved directly without knowledge of $c_s^2(r)$. However, we can already say a lot about the solution by inspection.

+ this is a Sturm-Liouville problem with

$$p(r) = r^2$$
$$q(r) = -\ell(\ell+1)$$
$$w(r) = \frac{r^2}{c_s^2(r)} \quad \leftarrow \text{weight function}$$
$$\lambda = \omega^2$$

Let's take the simplest possible example, that of constant sound-speed $c_s(r)$. Then

$$\frac{d}{dr}\left(r^2\frac{dA}{dr}\right) - \ell(\ell+1)A = -\frac{\omega^2}{c_s^2}r^2A$$

This is actually the equation for a spherical Bessel function. Indeed, spherical Bessel functions satisfy

$$\frac{d}{dx}\left(x^2\frac{dA}{dx}\right) - \ell(\ell+1)A + x^2A = 0. \quad \Rightarrow \text{as above if } x = \frac{\omega}{c}r$$

$$A_\ell(x) = \begin{Bmatrix} j_\ell(x) \\ y_\ell(x) \end{Bmatrix}.$$

The $y_\ell(x)$ functions are singular at $x=0$ $\rightarrow$ discard.

At the surface, $A_\ell(x) = 0 \Rightarrow A_\ell\left(\frac{\omega}{c}R_*\right) = 0$.

$\Rightarrow$ this means that $\frac{\omega}{c}R_*$ are zeros of the Bessel $j_\ell(x)$ function. There is an infinite number of them, noted $z_{\ell n}$ (the n-th zero of the $j_\ell(x)$ function)

$$\Rightarrow \quad \omega_{n\ell} = \frac{z_{\ell n}}{R_*}c_s.$$

$\Rightarrow$ Finally, putting everything together we have

$$p(r,\theta,\phi,t) = \sum_{n,\ell,m}' A_{\ell n}(r)B_\ell^m(\theta)C_m(\phi)D_{n\ell m}(t) \quad \text{where}$$

$$A_{\ell n}(r) = j_\ell\left(z_{\ell n}\frac{r}{R_*}\right)$$

$$C_m(\phi)B_\ell^m(\theta) = Y_\ell^m(\theta,\phi) \leftarrow \text{spherical harmonic}$$

$$D_{n\ell m}(t) = a_{n\ell m}\cos(\omega_{\ell n}t) + b_{n\ell m}\sin(\omega_{\ell n}t)$$

where $\omega_{\ell n} = \frac{z_{\ell n}}{R_*}c_s$