6.7 Non-homogeneous equations; introduction to Green's functions

6.7.1 Non homogeneous (regular) SL problems (ODEs)

Given the ODE $\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = F(x)$

with bcs $\begin{cases} \alpha u(a) + \beta u'(a) = 0 \\ \gamma u(b) + \delta u'(b) = 0 \end{cases}$

① Seek solutions of the homogeneous eigenvalue eq.

$$\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = -\lambda u$$

→ this yields the eigenfunctions $\{v_n\}$ and eigenvalues $\{\lambda_n\}$

② Write $F(x) = \sum_n b_n v_n(x)$

(with $b_n = \int_a^b r(x')F(x')v_n(x')\,dx'$, if the v_ns are properly normalized)

Then since we know that the solution can also be written as

$$u(x) = \sum_n a_n v_n(x)$$

we can write

$$\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = \sum_n -\lambda_n a_n v_n(x) = \sum_n b_n v_n(x)$$

and by identification, $a_n = -\frac{b_n}{\lambda_n}$

$$\Rightarrow \quad u(x) = \sum_n -\frac{b_n}{\lambda_n} v_n(x) = -\sum_n \int_a^b \frac{r(x')}{\lambda_n} F(x')v_n(x')v_n(x)\,dx'$$

$$= \int_a^b G(x;x')F(x')\,dx'$$

where $$G(x;x') = -\sum_n \frac{1}{\lambda_n} v_n(x') v_n(x)\, r(x')$$

- $G(x;x')$ is called the Green's function of the S.L problem

- It only depends on the characteristics of the homogeneous problem ($\{v_n\}$, $\{\lambda_n\}$) but, when integrated through with the forcing term $F(x)$, yields the solution of the forced problem

- Note that if the $\{v_n\}$ are not normalized then
$$G(x;x') = -\sum_n \frac{1}{\|v_n\|^2} \frac{r(x')}{\lambda_n} v_n(x') v_n(x) .$$
where $\|v_n\|^2 = \int_a^b r(x) v_n(x)^2 \, dx$.

<u>Example</u> Consider

$$y'' + y = 3\sin(2\pi x) \qquad \begin{matrix} y(0) = 0 \\ y(1) = 0 \end{matrix}$$

We seek the eigenfunctions of $y'' + y = -\lambda y$

$\rightarrow$ $$y'' + (1+\lambda) y = 0$$

so $$y = \alpha \cos(\sqrt{1+\lambda}\, x) + \beta \sin(\sqrt{1+\lambda}\, x)$$

with $\begin{cases} \alpha = 0 \\ \sqrt{1+\lambda_n} = n\pi \end{cases}$

$\Rightarrow \begin{cases} \lambda_n = n^2\pi^2 - 1 \\ v_n(x) = \sin(n\pi x) \end{cases}$

$\Rightarrow$ The Green's function $$G(x,x') = -\sum_n \frac{\sin(n\pi x)\sin(n\pi x')}{\lambda_n \|\sin n\pi x\|^2}$$
$$= -\sum_n \frac{2}{n^2\pi^2 - 1} \sin(n\pi x)\sin(n\pi x')$$

so that

$$\sum_n{}' \dot{a}_n(t) v_n(x) + \sum_n{}' \lambda_n a_n(t) v_n(x) = \sum_n b_n(t) v_n(x)$$

and (by orthogonality):

$$\dot{a}_n + \lambda_n a_n = b_n(t)$$

↳ integrating factor method:

$$\frac{d}{dt}\left(a_n e^{\lambda_n t}\right) = b_n(t) e^{\lambda_n t}$$

so $$a_n(t) e^{\lambda_n t} - a_n(0) = \int_0^t b_n(t') e^{\lambda_n t'} dt'$$

$$\Rightarrow a_n(t) = a_n(0) e^{-\lambda_n t} + e^{-\lambda_n t} \int_0^t b_n(t') e^{\lambda_n t'} dt'.$$

Putting it all together we find that

$$u(x,t) = \sum_n{}' a_n(0) e^{-\lambda_n t} v_n(x) + \int_0^t \sum_n{}' e^{-\lambda_n(t-t')} v_n(x)\, b_n(t')\, dt'$$

$$= \sum_n{}' a_n(0) e^{-\lambda_n t} v_n(x) + \int_0^t \int_a^b \sum_n{}' e^{-\lambda_n(t-t')} v_n(x) v_n(x') r(x') \; F(x',t')\, dx' dt'$$

So we can write

$$u(x,t) = \sum_n a_n(0) e^{-\lambda_n t} v_n(x) + \int_0^t \int_a^b G(x,t;x',t') F(x',t')\, dx' dt'$$

$$\text{with } G(x,t;x',t') = \sum_n{}' e^{-\lambda_n(t-t')} v_n(x) v_n(x') r(x')$$

Here G is called the Heat Equation Kernel.
↳ another example of a Green's function.

↳ u is the sum of
- the solution to the problem with no forcing
- \+ the weighted integral of $F(x,t)$ with the Green's function.

so the solution to the problem is $y(x) = \int_0^1 G(x,x')F(x')\,dx'$

$$y(x) = \int_0^1 \sum_n \frac{3\sin(2\pi x')}{1-n^2\pi^2} 2\sin(n\pi x)\sin(n\pi x')\,dx'$$

$$= \frac{3}{1-4\pi^2}\sin(2\pi x)$$

6.7.2 Application to parabolic/hyperbolic PDEs.

Now consider either $u_t - \frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = F(x,t)$

or $u_{tt} - \frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] = F(x,t)$.

Idea: Solve the associated Sturm-Liouville problem

$$\frac{1}{r(x)}\left[(p(x)u')' + q(x)u\right] + \lambda u = 0$$

to find the eigenvalues and eigenfunctions $\{v_n\}$, $\{\lambda_n\}$

then expand

$$F(x,t) = \sum_n b_n(t) v_n(x)$$

(in this case, $b_n(t) = \int_a^b F(x,t)\, r(x) v_n(x)\,dx$)

Assume a solution of the form

$$u(x,t) = \sum_n a_n(t) v_n(x)$$

and try the ansatz into the equation:

Parabolic case: $\sum_n \dot{a}_n(t) v_n(x) - \frac{1}{r(x)}\left[\left(p(x)\sum_n a_n(t) v_n'(x)\right)' + q(x)\sum a_n(t) v_n(x)\right]$

$$= \sum b_n(t) v_n(x)$$

Example of the drunks exiting the pub.

Recall:
$$\frac{\partial p}{\partial t} = k\frac{\partial^2 p}{\partial x^2} + S(x,t)$$

$$\begin{cases} p(x,0)=0 \\ \frac{\partial p}{\partial x} = 0 \quad \text{at } x = 0, L \\ S(x,t) = S_0 e^{-\frac{t}{c}}\,\delta\left(x-\frac{L}{2}\right) \quad \text{for } t>0 \end{cases}$$

(take $t_0=0$).

Homogeneous problem; separation of variables to get spatial eigenmodes $\Rightarrow$

$$\begin{cases} V_n(x) = \cos\left(\frac{n\pi x}{L}\right) \\ \lambda_n = \frac{n^2\pi^2}{L^2} \end{cases}$$

So, by the previous calculation, we have

$$u(x,t) = \sum_{n=0}^{\infty} a_n(0)\, e^{-\lambda_n t}\, V_n(x) + \int_0^t \int_0^L S(x',t')\, G(x,x';t,t')\, dx'dt'$$

where $a_n(0)$ are obtained by fitting u to initial conditions

$$u(x,0) = \sum_{n=0}^{\infty} a_n(0) V_n(x) = 0 \Rightarrow a_n(0) = 0$$

and where
$$G(x,x';t,t') = \sum_{n=0}^{\infty} e^{-\lambda_n(t-t')} \frac{V_n(x')\,V_n(x)\,r(x)}{\|V_n^2\|}$$
but $r(x)=1$

$$\Rightarrow \quad u(x,t) = \int_0^t\int_0^L \sum_{n=0}^{\infty} S_0 e^{-\frac{t'}{c}}\,\delta\left(x'-\frac{L}{2}\right) e^{-\frac{n^2\pi^2}{L^2}(t-t')} \frac{1}{\|V_n^2\|}\cos\left(\frac{n\pi x'}{L}\right)\cos\left(\frac{n\pi x}{L}\right) dx'dt'$$

$$= \int_0^t \sum_{n=0}^{\infty} S_0 e^{-\frac{t'}{c}} e^{-\frac{n^2\pi^2}{L^2}(t-t')} \cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{L}\right) dt' \cdot \frac{1}{\|V_n^2\|}$$

$$= \sum_{n=0}^{\infty} S_0 \cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi x}{L}\right) \frac{1}{\frac{n^2\pi^2}{L^2} - \frac{1}{c}} \left[e^{-\frac{t}{c}} - e^{-\frac{n^2\pi^2}{L^2}t}\right] \frac{1}{\|V_n^2\|}$$