(6) It is possible to construct a set of eigenfunctions $\{v_n\}$ of a regular Sturm-Liouville problem in such a way that

+ all eigenfunctions in the set are real
+ they are orthonormal w.r.t the inner product

$$\langle v_n, v_m \rangle = \int_a^b v_n(x) v_m(x) \, r(x) \, dx$$

+ the set is a <u>complete basis</u> for all piecewise continuous functions defined on the interval $[a,b]$, so that these functions can be written as the convergent series

$$f(x) = \sum_{n=0}^{\infty} a_n v_n(x) \qquad \forall x \in [a,b].$$

with

$$a_n = \int_a^b f(x) v_n(x) r(x) \, dx.$$

(note: if v_n are not normalized, then

$$a_n = \frac{\int_a^b f(x) v_n(x) r(x) dx}{\int_a^b v_n^2(x) r(x) dx}.$$

)

$\Rightarrow$ <u>Generalized Fourier Series</u>

<u>Examples</u>

<u>Example (A)</u> The Fourier functions.

let
$$\begin{cases} \dfrac{d^2u}{dx^2} = -\lambda u \\ u(0) = u(L) = 0 \end{cases}$$

We know that this is a Sturm Liouville problem (regular) with $p(x) = 1$, $r(x) = 1$, $q(x) = 0$.

The eigenvectors and eigenvalues are

$$\begin{cases} V_n(x) = \sin\left(\frac{n\pi x}{L}\right) \\ \lambda_n = \frac{n^2\pi^2}{L^2} \end{cases} \quad (n=1\ldots\infty)$$

let's redefine $m = n-1$ so that $\begin{cases} \lambda_m = \frac{(m+1)^2\pi^2}{L^2} \quad (m=0\ldots\infty) \\ V_m(x) = \sin\left(\frac{(m+1)\pi x}{L}\right) \end{cases}$

Let's verify each of the properties again:

(a) <u>Symmetry of the operator</u>

$$\int_0^L \left(\frac{d^2u}{dx^2}v - \frac{d^2v}{dx^2}u\right)dx$$ for u, v satisfying the same bcs

$$= \left[v\frac{du}{dx}\right] - \int_0^L \frac{du}{dx}\frac{dv}{dx}dx$$

$$- \left[u\frac{dv}{dx}\right] + \int_0^L \frac{du}{dx}\frac{dv}{dx}dx$$

$$= \left[v\frac{du}{dx} - u\frac{dv}{dx}\right]_0^L = 0$$

(b) <u>Orthogonality of the eigenfunctions</u>

assume $n \neq m$: $$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \int_0^L\left[\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right)\right]\frac{dx}{2}$$

$$= \frac{L}{(n-m)\pi}\sin\left(\frac{(n-m)\pi x}{L}\right)\Big]_0^L - \frac{L}{(n+m)\pi}\sin\left(\frac{(n+m)\pi x}{L}\right)\Big]_0^L$$

$$= 0$$

if $n=m$ then $$\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx = \int_0^L\left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right)\frac{dx}{2}$$

$$= \frac{L}{2}$$

(c) λ is indeed real

(e) $\{\lambda_n\}$ form an infinite monotonous sequence

(f) Every function f satisfying $f(0) = f(L) = 0$ can be written as

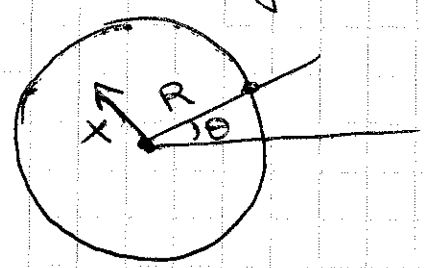
$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

with $$a_n = \frac{\langle f(x), \sin\left(\frac{n\pi x}{L}\right)\rangle}{\langle \sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\rangle}$$

$$= \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$ $\to$ see previous chapter.

<u>Example (B)</u> Bessel functions.

Let's consider a circular drum, assume it is oscillating in an axisymmetric way:



$$u_{tt} = \frac{c^2}{x}\frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x}\right)$$

$$u(R,t) = 0$$
$$|u(0,t)| < +\infty$$

At time $t=0$, we hit the drum dead center with a stick, giving it a velocity $u_t(x,0) = e^{-x^2/2\sigma^2}$

$$\begin{cases} u(x,0) = 0 \\ u_t(x,0) = e^{-x^2/2\sigma^2} \end{cases}$$

Separation of variables: $u(x,t) = A(x)B(t)$

$$\begin{cases} \frac{d^2B}{dt^2} = -c^2\lambda B \\ \frac{1}{x}\frac{d}{dx}\left(x\frac{dA}{dx}\right) = -\lambda A \end{cases}$$

The second equation represents a singular Sturm–Liouville problem with

$$\begin{cases} p(x) = x \\ q(x) = 0 \\ r(x) = x \end{cases}$$

Let the solutions be $A_n(x)$ with eigenvalues $\{\lambda_n\}$. Then, the associated $B_n(t)$ is

$$B_n(t) = \alpha_n \cos\left(c\sqrt{\lambda_n}\, t\right) + \beta_n \sin\left(c\sqrt{\lambda_n}\, t\right)$$

Given that $u(x,0) = 0 \Rightarrow \alpha_n = 0$

So

$$u(x,t) = \sum_{n=0}^{\infty} a_n \sin\left(c\sqrt{\lambda_n}\, t\right) A_n(x).$$

and to find the a_n, use the last IC:

$$u_t(x,0) = e^{-x^2/2\sigma^2}$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n c\sqrt{\lambda_n}\, A_n(x) = e^{-x^2/2\sigma^2}$$

Since the $\{A_n(x)\}$ form an orthogonal basis on $[0,R]$ with the weight function $r(x) = x$, we know that

$$a_n c\sqrt{\lambda_n} = \frac{\langle e^{-x^2/2}, A_n \rangle}{\langle A_n, A_n \rangle}$$

with $\langle u, v \rangle = \int_0^R x\, u(x) v(x)\, dx.$

<u>What are the $\{A_n(x)\}$ and the $\{\lambda_n\}$?</u>

Note that

$$\frac{d}{dx}\left(x \frac{dA}{dx}\right) = -\lambda A x \iff y^2 \frac{d^2A}{dy^2} + y\frac{dA}{dy} + y^2 A = 0$$

with $y = \sqrt{\lambda}\, x$

This is an equation for the 0th-order Bessel function

$$A_n(y) = \begin{Bmatrix} J_0(y) \\ Y_0(y) \end{Bmatrix}$$

(see attached handout on Bessel functions)

$$\Rightarrow \quad A_n(x) = \gamma_n J_0(\sqrt{\lambda_n}\, x) + \delta_n Y_0(\sqrt{\lambda_n}\, x)$$

To guarantee regularity at the origin $\delta_n = 0$.

To guarantee $A_n(R) = 0 \Rightarrow J_0(\sqrt{\lambda_n}\, R) = 0$ which implies

$$\sqrt{\lambda_n}\, R = z_n$$

where z_n is the n-th zero of J_0

$$\Rightarrow \quad \lambda_n = \frac{z_n^2}{R^2}$$

So finally,

$$u(x,t) = \sum a_n \sin\left(c \frac{z_n}{R} t\right) J_0\left(\frac{z_n}{R} x\right)$$

where

$$a_n = \frac{R}{c z_n} \frac{\int_0^R x e^{-x^2/2\sigma^2} J_0\left(\frac{z_n}{R} x\right) dx}{\int_0^R x J_0^2\left(\frac{z_n}{R} x\right) dx}$$

(see Maple file)

(7) Define

$$\mathcal{R}(u) = -\frac{\int_a^b u \mathcal{L}(u)\, dx}{\int_a^b r u^2\, dx}$$

(the Rayleigh Quotient)

then, the following theorem holds,

Theorem: • The principal eigenvalue λ_0 of a regular Sturm-Liouville problem is the solution of

$$\lambda_0 = \inf_{u \in V} R(u) \qquad \text{(Rayleigh-Ritz formula)}$$

where V is the space of all continuous & differentiable functions on (a,b) such that u satisfy the BCs of the Sturm-Liouville problem, and $u \neq 0$ (not the trivial function)

• The function u_0 for which the minimum of $R(u)$ is achieved is the corresponding eigenfunction of the principal eigenvalue

Proof: let $\{\lambda_0 \dots \lambda_n \dots\}$ be the set of all eigenvalues of the S.L problem with $\{v_0 \dots v_n \dots\}$ the set of corresponding orthonormal eigenfunctions

then $$u = \sum_n{}' a_n v_n(x)$$

and $$\mathcal{L}(u) = -\sum_n{}' a_n \lambda_n r(x) v_n(x).$$

Then $$\int_a^b u\mathcal{L}(u)\,dx = \int_a^b -\sum_n{}'\sum_m{}' a_n a_m \lambda_n r(x) v_n(x) v_m(x)\,dx$$

modulo some arguments about exchanging $\sum$ and $\int$

$$= -\sum_n{}'\sum_m{}' \int a_n a_m \lambda_n r(x) v_n(x) v_m(x)\,dx$$

$$= -\sum_n{}' a_n^2 \lambda_n$$

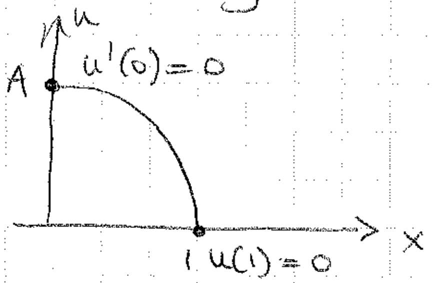
$$\int_a^b r(x) u^2\,dx = \int_a^b \sum_n{}'\sum_m{}' a_n a_m v_n(x) v_m(x) r(x)\,dx$$

same $$= \sum_n{}' a_n^2$$

$$R(u) = \frac{\int_0^1 u'^2 + x^2u^2 \, dx - \overbrace{\cancel{\left[u(1)u'(1) - u(0)u'(0)\right]}}^{0}}{\int_0^1 u^2 \, dx}$$

so we see that $R(u) \geq 0 \Rightarrow \lambda_0 \geq 0$

To obtain an estimate for λ_0, consider a trial function $u(x)$ that satisfy the boundary conditions



→ could try $u(x) = A\cos\left(\frac{\pi}{2}x\right)$

or $u(x) = A(1-x^2)$

Using the 2nd option

$$R(u) = \frac{\int_0^1 \cancel{A^2}(-2x)^2 + x^2\cancel{A^2}(1-x^2)^2 \, dx}{\int_0^1 \cancel{A^2}(1-x^2)^2 \, dx}$$

$$= \frac{\int_0^1 (4x^2 + x^2 - 2x^4 + x^6) \, dx}{\int_0^1 1 - 2x^2 + x^4 \, dx}$$

$$= \frac{\frac{5}{3} - \frac{2}{5} + \frac{1}{7}}{1 - \frac{2}{3} + \frac{1}{5}} = \frac{37}{14} \Rightarrow 0 \leq \lambda_0 \leq \frac{37}{14}$$

Exercise; try the same procedure with $A\cos\left(\frac{\pi}{2}x\right)$

<u>Note</u>: True solution has $\lambda_0 = 2.597\ldots$ $\quad \frac{37}{14} = 2.64$

(2) It is actually possible to show that the sequence of eigenvalues of the S-L problem $(pu_n')' + qu_n = -\lambda_n r u_n$ is also the set of <u>all stationary pts</u> of the Rayleigh quotient $R(u)$ over V, and the eigenfunctions are the functions for which this stationary pt is achieved:

$$\lambda_n = R(v_n)$$

This form is often v. useful to determine the sign of λ_0, and to obtain order-of-magnitude estimates for it

(6.1) Example 1. Consider the S.L problem

$$\begin{cases} u'' + \lambda u = 0 \\ u(0) - u'(0) = 0 \qquad\qquad x \in [0,1] \\ u(1) + u'(1) = 0 \end{cases}$$

Here, we have a S.L problem with

$$\begin{cases} p(x) = 1 \\ q(x) = 0 \\ r(x) = 1 \end{cases}$$

so $$R(u) = \frac{\int_0^1 u'^2\,dx - [u(1)u'(1) - u(0)u'(0)]}{\int_0^1 u^2\,dx}$$

but $u'(1) = -u(1)$ and $u'(0) = u(0) \Rightarrow [u(1)u'(1) - u(0)u'(0)] = -u(1)^2 - u(0)^2$

$\rightarrow$ so clearly $R(u) \geq 0$ for all u, which proves that $\lambda_0 \geq 0$.

(6.4) Example 2 Consider the S.L. problem.

$$\begin{cases} u'' + (\lambda - x^2) u = 0 \\ u'(0) = 0 \\ u(1) = 0 \end{cases}$$

We want to find an estimate for λ_0.

Here, we have the S.L problem with

$$\begin{cases} p(x) = 1 \\ q(x) = -x^2 \\ r(x) = 1 \end{cases} \qquad \text{so}$$

so $$R(u) = \frac{\sum_n a_n^2 \lambda_n}{\sum_n a_n^2}$$

now given that we know that $\forall n > 0$ $\lambda_n > \lambda_0$ then

$$R(u) \geq \frac{\lambda_0 \sum_n a_n^2}{\sum_n a_n^2} = \lambda_0$$

To have equality, we would require that $a_n = 0$ $\forall n > 0$ so that

$$R(u) = \frac{a_0^2 \lambda_0}{a_0^2} = \lambda_0$$

If $u_0 = a_0 v_0$ then u_0 is indeed the eigenfunction corresponding to the principle eigenvalue λ_0.

<u>Notes</u> ① given that $\int_a^b u\mathcal{L}(u)\,dx$

$$= \int_a^b u\left[(p(x)u')' + q(x)u\right]dx$$

$$= \int_a^b q(x)u^2\,dx + \left[up(x)u'\right]_a^b - \int_a^b p(x)u'^2\,dx$$

then $$R(u) = \inf_{u \in V}\left[\frac{\int_a^b (p(x)u'^2 - q(x)u^2)\,dx - [uu'p]_a^b}{\int_a^b ru^2\,dx}\right]$$

This becomes particularly simple for homogeneous Neumann or Dirichlet conditions : in that case

$$R(u) = \inf_{u \in V}\left[\frac{\int_a^b [p(x)u'^2 - q(x)u^2]\,dx}{\int_a^b ru^2\,dx}\right]$$

(8) Consider a regular S.L problem with eigenvalues $\{\lambda_0 \dots \lambda_n \dots\}$ and eigenfunctions $\{v_0 \dots v_n \dots\}$.
Then v_n has exactly <u>n roots</u> over the interval (a,b)

In particular, v_0 has no node within (a,b)

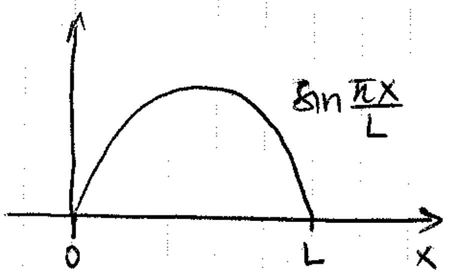
<u>Remark</u>: This is why the simplest guess for $y(x)$ for estimating λ_0 actually is also the best.

<u>Example</u>:

$$\begin{cases} u'' + \lambda u = 0 \\ u(0) = u(L) = 0 \end{cases}$$

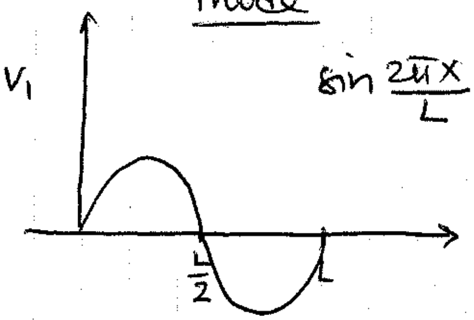
$$\begin{cases} \lambda_n = \dfrac{\pi^2}{L^2}(n+1)^2 \\ v_n = \sin\left(\dfrac{\pi x}{L}(n+1)\right) \end{cases} \quad n \geq 0$$

v_0:
0 nodos



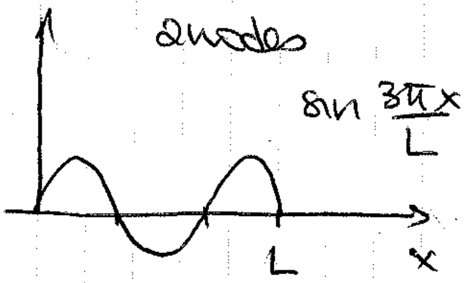
1 node

v_1



2 nodes

v_2



etc...

(8) Asymptotic ($n \to \infty$) approximations to the eigenfunctions and eigenvalues of a regular SL problem

For large n, we know that $\lambda_n \to +\infty$; if this is the case, it is possible to approximate the eigenfunctions by

$$v_n(x) \simeq \frac{1}{(r(x)p(x))^{1/4}} \left\{ \alpha \cos\left[\sqrt{\lambda_n} \int_a^x \sqrt{\frac{r(x')}{p(x')}}\, dx'\right] + \beta \sin\left[\sqrt{\lambda_n} \int_a^x \sqrt{\frac{r(x')}{p(x')}}\, dx'\right] \right\}$$

(This formula is derived from the WKB approximation (see AMS 212b))

In that case it's easy to see that

$$\sqrt{\lambda_n} \int_a^b \sqrt{\frac{r(x')}{p(x')}}\, dx' \simeq n\pi$$

$$\Rightarrow \qquad \lambda_n \simeq \left(\frac{n\pi}{\int_a^b \sqrt{\frac{r(x')}{p(x')}}\, dx'} \right)^2$$

<u>Example</u>

$$\begin{cases} u'' + \lambda u = 0 \\ u(0) = u'(0) \\ u(1) = -u'(1) \end{cases}$$

we saw that $\lambda \geq 0$

This time, let's look for the eigensolutions:

$$u(x) = A\cos\sqrt{\lambda}\, x + B\sin\sqrt{\lambda}\, x$$

to satisfy the bcs we calculate

$u'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$

so
$$\begin{cases} A = B\sqrt{\lambda} \\ A\cos\sqrt{\lambda} + B\sin\sqrt{\lambda} = +A\sqrt{\lambda}\sin\sqrt{\lambda} - B\sqrt{\lambda}\cos\sqrt{\lambda} \end{cases}$$

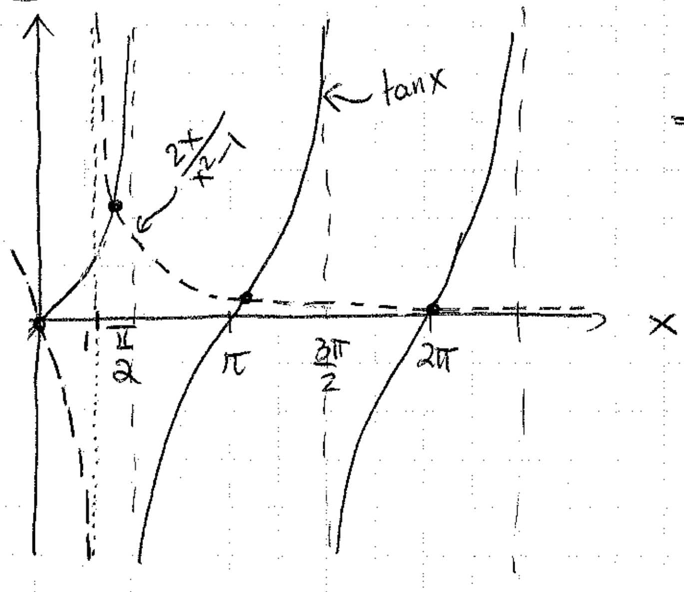
$$\Rightarrow \begin{cases} \sqrt{\lambda}\cos\sqrt{\lambda} + \sin\sqrt{\lambda} = +\lambda\sin\sqrt{\lambda} - \sqrt{\lambda}\cos\sqrt{\lambda}. \\ A = B\sqrt{\lambda} \end{cases}$$

$$\Rightarrow \begin{cases} 2\sqrt{\lambda}\cos\sqrt{\lambda} = (\lambda-1)\sin\sqrt{\lambda} \\ A = B\sqrt{\lambda} \end{cases}$$

$$\Rightarrow \begin{cases} \tan\sqrt{\lambda} = \dfrac{2\sqrt{\lambda}}{\lambda-1} \\ A = B\sqrt{\lambda} \end{cases}$$

to find λ, we must solve the equation $\tan x = \dfrac{2x}{x^2-1}$

Graphically with $x>0$



$\Rightarrow$ looks like

$x_n \simeq n\pi$ for large n

$\Rightarrow \lambda_n \simeq n^2\pi^2$

Check : using the asymptotic formula with
$r(x) = p(x) = 1 \quad q(x) = 0 \qquad \begin{matrix} a=0 \\ b=1 \end{matrix}$

$\Rightarrow \quad \lambda_n \cong (n^2\pi^2)$. indeed for large n.

6.6 Example of application: wave in a non-homogeneous medium

Consider the wave equation for varying wave speed:

$$\frac{\partial^2 u}{\partial t^2} = c^2(x)\frac{\partial^2 u}{\partial x^2} \qquad c^2(x) > 0 \quad \forall x \in [0,1].$$

on a finite string : $x \in [0,1]$.
with $u(0) = u(1) = 0 \quad \forall t$

Then this is an archetype S.L problem / eigenfunction expansion problem.

let $\quad u = T(t)F(x)$ then

$$\frac{\ddot{T}}{T} = -\lambda \qquad c^2(x)\frac{F''}{F} = -\lambda$$

so $\quad F'' = -\dfrac{\lambda F}{c^2(x)}$, a Sturm-Liouville problem with

$$\begin{cases} p(x) = 1 \\ q(x) = 0 \\ r(x) = \dfrac{1}{c^2(x)} \end{cases}$$

What can we learn from this without actually solving the equations?

① $\lambda \geq 0$.

Indeed: $\quad R(u) = \dfrac{\int_0^1 u'^2\,dx}{\int_0^1 \frac{u^2}{c^2(x)}\,dx} \geq 0$ for any function u.

② Some estimate of the fundamental mode of vibration can be made by minimizing $R(u)$: $\quad f_0 = \sqrt{\lambda_0}$ with

$$0 \leq \lambda_0 \leq \frac{\int_0^1 \bar{u}'^2\,dx}{\int_0^1 \frac{\bar{u}^2}{c^2(x)}\,dx}$$

(i.e. try $\bar{u} = \sin(\pi x)$ or $\bar{u} = x(1-x)$)

(3) Some estimate of the high frequencies of vibration can be made: $f_n \simeq \sqrt{\lambda_n}$ with

$$\lambda_n \simeq \left(\frac{n\pi}{\int_0^1 \frac{1}{c(x)} dx} \right)^2$$

<u>Example</u>: suppose we consider a string with a slight defect at $x = x_0$

$$c(x) = c_0\left(1 + \epsilon e^{-\frac{(x-x_0)^2}{2\sigma^2}}\right)$$

then

$$\frac{1}{c(x)} \simeq \frac{1}{c_0}\left(1 - \epsilon e^{-\frac{(x-x_0)^2}{2\sigma^2}}\right)$$

and for large n

$$\lambda_n \simeq n^2\pi^2 \left[\int_0^1 \frac{1}{c_0}\left(1 - \epsilon e^{-\frac{(x-x_0)^2}{2\sigma^2}}\right) dx \right]^{-2}$$

- if $\sigma \ll 1$ then the width of the Gaussian is small enough that we can approximate

$$\int_0^1 e^{-\frac{(x-x_0)^2}{2\sigma^2}} dx \simeq \int_{-\infty}^{+\infty} e^{-\frac{(x-x_0)^2}{2\sigma^2}} dx \simeq \sqrt{2\pi}\,\sigma$$

In that case $\lambda_n \simeq n^2\pi^2 c_0^2 \left(1 - \sqrt{2\pi}\,\sigma\epsilon\right)^{-2}$

$$\simeq n^2\pi^2 c_0^2 \left(1 + 2\sqrt{2\pi}\,\sigma\epsilon\right)$$

→ by comparing the frequencies "observed" on the imperfect string to those from a theoretical "perfect" string, we can deduce $\underline{\underline{\sigma\epsilon}}$ but not x_0.