$\Rightarrow$ The set of decoupled ODEs for the B_n are

$$\dot{B}_n + \frac{n^2\pi^2 k}{L^2} B_n = \frac{2}{L} S_0 e^{-(t-t_0)/\tau} \cos\left(\frac{n\pi}{2}\right)$$

$\rightarrow$ the general solution of the homogeneous problem is

$$B_n^G(t) = \alpha_n e^{-\frac{n^2\pi^2 k}{L^2} t}$$

The particular solution: try $B_n^{PS}(t) = K e^{-\frac{t-t_0}{\tau}}$

$$\Rightarrow \quad -\frac{1}{\tau} K + \frac{n^2\pi^2 k}{L^2} K = \frac{2}{L} S_0 \cos\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \quad K = \frac{\frac{2}{L} S_0 \cos\left(\frac{n\pi}{2}\right)}{\frac{n^2\pi^2 k}{L^2} - \frac{1}{\tau}}$$

so finally, we have $B_n(t) = \alpha_n e^{-\frac{n^2\pi^2 k}{L^2} t} + \dfrac{\frac{2}{L} S_0 \cos\left(\frac{n\pi}{2}\right)}{\frac{n^2\pi^2 k}{L^2} - \frac{1}{\tau}} e^{-\frac{(t-t_0)}{\tau}}$ for $(n \neq 0)$

$\Rightarrow$ $p(x,t) = \sum_{n=0}^{\infty} A_n(x) B_n(t)$ is the complete solution, where the α_ns remain to be determined.

At $t = t_0$ $p(x,t) = 0$ (the street is empty before $t = t_0$)

$$\Rightarrow \quad \sum_{n=0}^{\infty} A_n(x) B_n(0) = 0$$

$$b_0 - \frac{\tau}{L} S_0 e^{-\frac{t-t_0}{\tau}} + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{L}\right)\left[\alpha_n e^{-\frac{n^2\pi^2 k t_0}{L^2}} + \frac{\frac{2}{L} S_0 \cos\left(\frac{n\pi}{2}\right)}{\frac{n^2\pi^2 k}{L^2} - \frac{1}{\tau}}\right] = 0$$

$$\Rightarrow \quad \alpha_n = \frac{-S_0 \cos\left(\frac{n\pi}{2}\right) \cdot \frac{2}{L}}{\frac{n^2\pi^2 k}{L^2} - \frac{1}{\tau}} e^{\frac{n^2\pi^2 k t_0}{L^2}} \quad \text{and} \quad b_0 = \frac{\tau}{L} S_0$$

$$\Rightarrow p(x,t) = \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{L}\right) \frac{\frac{2}{L} S_0 \cos\left(\frac{n\pi}{2}\right)}{\frac{n^2\pi^2 k}{L^2} - \frac{1}{\tau}} \left(-e^{-\frac{n^2\pi^2}{L} k (t-t_0)} + e^{-(t-t_0)/\tau}\right)$$

$$+ \frac{\tau}{L} S_0 \left(1 - e^{-\frac{t-t_0}{\tau}}\right)$$

Note. ① The total number of people in the street at any time is easily derived from the PDE

$$\Rightarrow \quad \frac{\partial p}{\partial t} = k\frac{\partial^2 p}{\partial x^2} + S(x,t)$$

$$\hookrightarrow \frac{\partial}{\partial t}\int_0^L p(x,t)\,dx = k\int_0^L \frac{\partial^2 p}{\partial x^2}\,dx + \int_0^L S(x,t)\,dx$$

$$= k\left[\frac{\partial p}{\partial x}\right]_0^L + S_0 e^{-\frac{t-t_0}{\tau}}$$

$$= S_0 e^{-\frac{t-t_0}{\tau}}$$

So

$$\int_0^L p(x,t)\,dx = \int_{t_0}^{t} S_0 e^{-\frac{t'-t_0}{\tau}}\,dt'$$

$$= \tau S_0\left[1 - e^{-\frac{t-t_0}{\tau}}\right] \quad (t > t_0)$$

↳ at any time the # of people in the street is equal to the total # which has already left the pub as expected.

② See movies:

- if $\tau \ll \frac{L^2}{\pi^2 k}$ then a large # of people are rapidly released, and then diffuse away from pub entrance
- if $\tau \gg \frac{L^2}{\pi^2 k}$ then the diffusion is faster than release & the people are always $\sim$ evenly spread in the street.

③ Note the "resonnance" between $\frac{1}{\tau}$ and $\frac{n^2\pi^2 k}{L^2}$

$\Rightarrow$ if $\tau \ll \frac{L^2}{\pi k}$ then $\exists\, n$ such that $\frac{1}{\tau} \simeq \frac{n^2\pi^2 k}{L^2}$

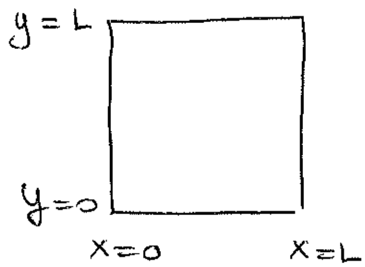
That n determines the typical initial "width" of the people density function. (See movie) as $\Delta = \frac{L}{n\pi}$

③ <u>Poisson equation</u>

Suppose we want to solve $\nabla^2 T = -H(x,y)$

to obtain the steady-state temperature profile in a metallic plate, heated as prescribed by $H(x,y)$ and with $T=0$ on all 4 sides; take $k=1$

↑ heating source



Note that the − sign comes from

$$\frac{\partial T}{\partial t} = \nabla^2 T + H(x,y)$$

→ in steady state $\nabla^2 T = -H$.

The spatial eigenmodes in x-direction are (see previous lectures), for $T(0,y) = T(L,y) = 0$

$$A_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

→ Assume $T(x,y) = \sum_{n=1}^{\infty} A_n(x) B_n(y)$

then

$$\sum_{n=1}^{\infty} -\frac{n^2\pi^2}{L^2} A_n(x) B_n(y) + A_n(x) \frac{d^2 B_n}{dy^2} = -H(x,y)$$

Noting that $\int_0^L A_n(x) A_m(x)\, dx = \frac{L}{2} \delta_{mn}$,

$$\frac{d^2 B_n}{dy^2} - \frac{n^2\pi^2}{L^2} B_n = -\frac{2}{L} \int_0^L H(x,y) \sin\left(\frac{n\pi x}{L}\right) dx.$$

$$= -h_n(y).$$

Suppose that $H(x,y) = \delta\left(x - \frac{L}{2}\right) \delta\left(y - \frac{L}{2}\right)$ to model a point source

Then $\dfrac{d^2 B_n}{dy^2} - \dfrac{n^2\pi^2}{L^2} B_n = -\sin\left(\dfrac{n\pi}{2}\right) \delta\left(y - \dfrac{L}{2}\right) \cdot \dfrac{2}{L}$

Laplace transforms, review

Laplace transforms are very useful for solving non-homogeneous linear ODES.

Idea: let f be a function of t

The Laplace transform of f is

$$\mathcal{L}(f) = \hat{f}(p) = \int_0^\infty f(t) e^{-pt}\, dt.$$

Properties: $\mathcal{L}(f') = p\hat{f}(p) - f(0)$

since $\displaystyle\int_0^\infty \frac{df}{dt} e^{-pt}\, dt = \left[f e^{-pt} \right]_0^\infty + \int_0^\infty p f e^{-pt}\, dt$

$= p\hat{f}(p) - f(0)$

$\mathcal{L}(f'') = p^2 \hat{f}(p) - p f(0) - f'(0)$

(proof is similar).

So given a linear ODE with constant coefficients

$$a f'' + b f' + c f = g(t) \qquad (*)$$

$$\mathcal{L}(*) \Rightarrow \quad a\left[p^2 \hat{f}(p) - p f(0) - f'(0) \right] + b\left[p\hat{f}(p) - f(0) \right] + c\hat{f}(p) = \int_0^\infty g(t) e^{-pt}\, dt = G(p)$$

Suppose $f(0)$ and $f'(0)$ are known (initial value problem) then this is an algebraic equation for $\hat{f}(p)$.

To recover $f(t)$, we need to do an inverse Laplace transform.

For detail on Inverse Laplace transforms, see handout. Usually, it's easy to find the solution using Inverse Laplace transform tables.

<u>Here</u>:

$$\frac{d^2B_n}{dy^2} - \frac{n^2\pi^2}{L^2} B_n = -\sin\left(\frac{n\pi}{2}\right) \delta\left(y-\frac{L}{2}\right)\cdot\frac{2}{L}$$

$$\Rightarrow \quad p^2 \hat{B}_n - p B_n(0) - B_n'(0)$$

$$- \frac{n^2\pi^2}{L^2} \hat{B}_n = -\frac{2}{L}\sin\left(\frac{n\pi}{2}\right)\int_0^\infty \delta\left(y-\frac{L}{2}\right)e^{-py}\,dy$$

$$= -\sin\left(\frac{n\pi}{2}\right)e^{-p\frac{L}{2}}\cdot\frac{2}{L}$$

$B_n(0)=0$ but $B_n'(0)$ is unknown. Let's leave it as is for the moment.

$$\Rightarrow \quad \hat{B}_n\left[p^2 - \frac{n^2\pi^2}{L^2}\right] = B_n'(0) - \frac{2}{L}\sin\left(\frac{n\pi}{2}\right)e^{-p\frac{L}{2}}$$

$$\text{so} \quad \hat{B}_n(p) = \frac{B_n'(0)}{p^2-\frac{n^2\pi^2}{L^2}} - \frac{2}{L}\sin\left(\frac{n\pi}{2}\right)e^{-p\frac{L}{2}}\frac{1}{p^2-\frac{n^2\pi^2}{L^2}}$$

From tables:

- The inverse transform of $\frac{1}{p^2-a^2}$ is $\frac{\sinh(ay)}{a}$
- The inverse transform of

$$\frac{e^{-pb}}{p^2-a^2} \text{ is } \begin{cases} \frac{\sinh(a(y-b))}{a} & \text{if } y>b \\ 0 & \text{if } 0<y<b \end{cases}$$

$$\Rightarrow \quad B_n(y) = \frac{B_n'(0)}{\frac{n\pi}{L}} \cdot \sinh\left[\frac{n\pi y}{L}\right]$$

$$- \frac{2}{L}\sin\left(\frac{n\pi}{2}\right) \sinh\left(\frac{n\pi(y-\frac{L}{2})}{L}\right)\frac{1}{\frac{n\pi}{L}} \quad \text{if } y > \frac{L}{2}$$

At $y = L$, the solution is such that $B_n(L) = 0 \Rightarrow$

$$B_n'(0) \sinh(n\pi) - \frac{2}{L}\sin\left(\frac{n\pi}{2}\right)\sinh\left(\frac{n\pi}{2}\right) = 0$$

$$\Rightarrow \quad B_n'(0) = \frac{2}{L}\,\frac{\sin\left(\frac{n\pi}{2}\right)\sinh\left(\frac{n\pi}{2}\right)}{\sinh(n\pi)}$$

So finally, we have

$$T(x,y) = \sum_{n=1}^{\infty}{}' \sin\left(\frac{n\pi x}{L}\right) \frac{\sin\left(\frac{n\pi}{2}\right)}{\frac{n\pi}{L}} \frac{2}{L}\left[\frac{\sinh\left(\frac{n\pi}{2}\right)}{\sinh(n\pi)}\sinh\left(\frac{n\pi y}{L}\right) - \sinh\left(\frac{n\pi(y-\frac{L}{2})}{L}\right) H\left(y-\frac{L}{2}\right)\right]$$

Heaviside function

<u>Note</u>: This expression is slightly awkward, but it can be shown that it indeed leads to the correct behavior in y, which should be symmetry across the $y = \frac{L}{2}$ line. (Homework!)