4.4 Linear, forced equations

We now consider forced linear PDES of the kind

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = F(x,t) \qquad \text{(Forced Wave Equation)}$$

$$\frac{\partial u}{\partial t} - k \frac{\partial^2 u}{\partial x^2} = F(x,t) \qquad \text{(Forced heat Equation)}$$

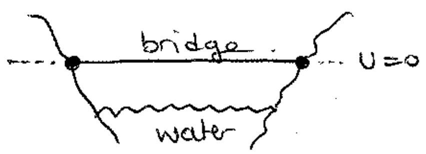
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = F(x,y) \qquad \text{("Forced Laplace equation" = Poisson equation)}$$

The method for solving these problems will be illustrated through examples.

① Forced wave equation

Example : A bridge, suspended, and the wind forcing (cf Tacoma Narrows)
(in 2D : a metal plate, with some sand on it, and a speakerphone nearby ; see Exploratorium).

let
$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = F(x,t) & \\ u(x,t=0) = 0 & u(0,t) = 0 \\ u_t(x,t=0) = 0 & u(L,t) = 0 \end{cases}$$



→ A forced string, pinned at the sides, initially at rest.

<u>Method</u>: 1. Find the <u>spatial</u> eigenmodes $A_n(x)$ homogeneous problem - with same bcs

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 .$$

These will generally be mixtures of sines and cosines.

Here (see previous lectures)

$$A_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

2. Assume that the full solution can be written as

$$u(x,t) = \sum_{n=0}^{\infty} A_n(x) B_n(t)$$

and plug into PDE

$$\Rightarrow \sum_{n=0}^{\infty} A_n(x) \frac{d^2 B_n}{dt^2} - c^2 B_n(t) \frac{d^2 A_n}{dx^2} = F(x,t)$$

$$\Rightarrow \sum_{n=0}^{\infty} A_n(x) \frac{d^2 B_n}{dt^2} + \frac{c^2 n^2 \pi^2}{L^2} B_n(t) A_n(x) = F(x,t). \quad (*)$$

3. Note that $\displaystyle\int_0^L A_n(x) A_m(x)\, dx = 0 \quad \forall n \neq m$

$\qquad = \frac{L}{2}$ if $n = m$

Aside: The orthogonality property of the eigenfunctions will be true for a wide class of problems, see next chapter.

So take $(*)$ and multiply by $A_m(x)$, then integrate over $[0, L]$

$$\Rightarrow \frac{L}{2} \frac{d^2 B_m}{dt^2} + \frac{c^2 m^2 \pi^2}{L^2} \cdot \frac{L}{2} B_m = \int_0^L F(x,t) \sin\left(\frac{m\pi x}{L}\right) dx.$$

$$\Rightarrow \ddot{B}_m + \frac{c^2 m^2 \pi^2}{L^2} B_m = \frac{2}{L} \int_0^L F(x,t) \sin\left(\frac{m\pi x}{L}\right) dx$$

$$= f_m(t).$$

$\Rightarrow$ we now get a set of independent ODEs, one for each value of m. These are forced, second-order linear ODEs. (which you should be able to solve...)

Simple example

Suppose $F(x,t) = \sin\left(\frac{2\pi x}{L}\right)\cos(\omega t)$

then $f_m(t) = \int_0^L \frac{2}{L}\sin\left(\frac{2\pi x}{L}\right)\cos(\omega t)\sin\left(\frac{m\pi x}{L}\right)dx$

$$= \begin{cases} \frac{2}{L}\cdot\frac{L}{2}\cdot\cos\omega t & \text{if } m=2 \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ we have 2 types of ODES to solve:

$$\ddot{B}_2 + \frac{4c^2\pi^2}{L^2}B_2 = \cos\omega t$$

and $$\ddot{B}_m + \frac{c^2m^2\pi^2}{L^2}B_m = 0 \qquad \forall\, m\neq 2.$$

For all of them, the solution to the homogeneous equation is

$$B_m(t) = \alpha_m\cos\left(\frac{cm\pi}{L}t\right) + \beta_m\sin\left(\frac{m\pi ct}{L}\right)$$

For the $B_2(t)$ function, we have to add a particular solution to the forced problem: here try

$$B_2^{PS}(t) = K\cos\omega t$$

a constant to be determined by plugging into equation

$$\Rightarrow \quad -K\omega^2 + \frac{4c^2\pi^2}{L^2}K = 1$$

$$\Rightarrow \quad K = \frac{1}{\frac{4c^2\pi^2}{L^2} - \omega^2}$$

$$\Rightarrow \quad B_2(t) = \alpha_2\cos\left(\frac{2c\pi t}{L}\right) + \beta_2\sin\left(\frac{2c\pi t}{L}\right) + \frac{\cos(\omega t)}{\frac{4c^2\pi^2}{L^2} - \omega}$$

$\Rightarrow$ So finally, we have

$$u(x,t) = \sum_{n=1}^{\infty} A_n(x)\, B_n(t) \qquad \text{where} \quad A_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

$$B_n(t) = \alpha_n \cos\left(\frac{n\pi c t}{L}\right) + \beta_n \sin\left(\frac{n\pi c t}{L}\right) + \frac{\cos \pi t}{\frac{4c^2\pi^2}{L^2} - \omega^2}\, \delta_{n,2}$$

To find the arbitrary constants α_n, β_n, we fit the initial conditions:

$u(x,0) = 0$

$$\Rightarrow \sum_{n=1}^{\infty} A_n(x) B_n(0) = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} A_n(x) \left[\alpha_n + \frac{1}{\frac{4c^2\pi^2}{L^2} - \omega^2}\, \delta_{n,2} \right] = 0$$

$$\Rightarrow \alpha_n + \frac{1}{\frac{4c^2\pi^2}{L^2} - \omega^2}\, \delta_{n,2} = 0$$

$$\Rightarrow \begin{cases} \alpha_n = 0 & \text{if } n \neq 2 \\ \alpha_2 = -\dfrac{1}{\frac{4c^2\pi^2}{L^2} - \omega^2} \end{cases}$$

$$u_t(x,0) = 0 \Rightarrow \sum_{n=1}^{\infty} A_n(x)\, \dot{B}_n(0) = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} A_n(x) \left[\beta_n \frac{n\pi c}{L} \right] = 0$$

$$\Rightarrow \beta_n = 0 \quad \forall n.$$

So: the solution becomes very simple!

$$u(x,t) = \left[-\frac{1}{\frac{4c^2\pi^2}{L^2} - \omega^2} \cdot \cos\left(\frac{2\pi c t}{L}\right) + \frac{\cos \omega t}{\frac{4c^2\pi^2}{L^2} - \omega^2} \right] \sin\left(\frac{2\pi x}{L}\right)$$

<u>We note</u>: • $F(x,t) = \cos \omega t \sin\left(\frac{2\pi x}{L}\right)$

specifically forces the system in one of its <u>spatial eigenmodes</u>

→ then this eigenmode is the only one to be excited (see solution).

If $F(x,t)$ had a more complex spatial structure, other modes would be excited too.

• The solution responds by oscillating at <u>two</u> different frequencies simultaneously:

Beating phenomenon. $\Leftarrow$ $\left\{ \begin{array}{l} \text{– at the } \underline{\text{forcing}} \text{ frequency} \\ \text{– at the } \underline{\text{intrinsic}} \text{ frequency of the eigenmode.} \end{array} \right.$

• The amplitude of the response goes like

$$\frac{1}{\frac{4c^2\pi^2}{L^2} - \omega^2}$$

intrinsic frequency squared

forcing frequency square

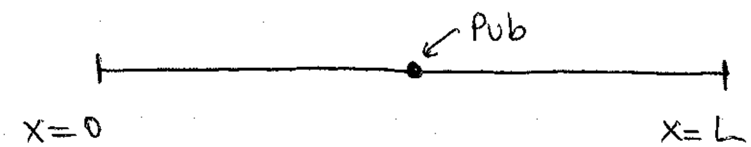
→ if the forcing frequency approaches the intrinsic frequency then the amplitude of the response can be <u>huge</u>

This phenomenon is called <u>resonance</u>.

<u>Note</u>: That doesn't mean the amplitude can ever be ∞: instead if $\omega = \frac{4c^2\pi^2}{L^2}$, the amplitude of the mode grows linearly with time. (Homework: prove this).

② Forced diffusion equation

A pub in England rings last orders at 11:00 pm, at which point people start to leave and go back home. They are only "locals", i.e, people living in the same 1D street. Being quite drunk, they walk randomly in the street although they dont leave it. We assume they can't find their keys and stay in the street a long time ...



← Street of length L.

Pub at $x=\frac{L}{2}$

⊕ We model the evolution of the population density in the street as a diffusion process:

⟹
$$\frac{\partial p}{\partial t} = k\frac{\partial^2 p}{\partial x^2} + S(x,t)$$

- population density $p=p(x,t)$.
- $S(x,t)=$ # of people / unit time being released in the street by the pub.

⊕ At $t=t_0$ the street is empty

⊕ To model the "they don't leave the street" idea, we use insulating boundary conditions.

$$\Rightarrow \quad \frac{\partial p}{\partial x} = 0 \quad \text{at both boundaries.}$$

⊕ To model the flux of people out of the pub, we assume

$$S(x,t) = S_0 e^{-(t-t_0)/\tau}\,\delta\left(x-\frac{L}{2}\right)$$

$t_0 =$ last orders time

$\tau \simeq$ time till closing, say 1/2 hour.

$\delta =$ a delta function

Recall: $\int_{-\infty}^{\infty} \delta(x-a) f(x)\,dx = f(a)$.

Solution. 1. Find the spatial eigenmodes of the homogeneous problem.

- From previous lectures, we know that

$$\begin{cases} A_0(x) = a_0 x + b_0 \\ A_n(x) = a_n \cos(\lambda_n x) + b_n \sin(\lambda_n x) \end{cases} \qquad (\lambda \text{ to be determined})$$

to satisfy $\frac{dp}{dx} = 0$ at both ends we need

- $n \neq 0$ $\quad \frac{dA_n}{dx} = \lambda_n\left(-a_n \sin(\lambda_n x) + b_n \cos(\lambda_n x)\right)$

$$\Rightarrow \begin{cases} \left.\frac{dA_n}{dx}\right|_{x=0} = 0 \Rightarrow b_n = 0 \\ \left.\frac{dA_n}{dx}\right|_{x=L} = 0 \Rightarrow \lambda_n = \frac{n\pi}{L} \end{cases} \Rightarrow A_n(x) = \cos\left(\frac{n\pi x}{L}\right)$$

(ignore constant)

- $n = 0$ $\quad A_0(x) = \text{constant} = 1$ (ignore constant).
  ↳ can be written as $\cos\left(\frac{0\pi x}{L}\right)$.

3. Note that $\int_0^L A_n(x) A_m(x)\, dx = \frac{L}{2}\delta_{mn} + \frac{L}{2}\delta_{m0}\delta_{mn}$

2. Suppose the solution is

$$p(x,t) = \sum_0^\infty A_n(x) B_n(t) \quad \text{and plug into PDE}$$

$$\Rightarrow \sum_0^\infty A_n(x)\dot{B}_n(t) = k\sum_0^\infty -\frac{n^2\pi^2}{L^2} A_n(x) B_n(t) + S(x,t)$$

multiply by $A_m(x)$, integrate in $[0,L]$ ...

- $m \neq 0$ $\quad \frac{L}{2}\dot{B}_m(t) = -\frac{m^2\pi^2 k}{L^2}\cdot\frac{L}{2} B_m(t) + \int_0^L S(x,t) A_m(x)\, dx$

Now $\int_0^L A_m(x) S_0 e^{-(t-t_0)/\tau}\, \delta(x - \frac{L}{2})\, dx$

$$= S_0 e^{-(t-t_0)/\tau} A_m\left(\tfrac{L}{2}\right) = S_0 e^{-(t-t_0)/\tau} \cos\left(\frac{m\pi}{2}\right)$$

- $m = 0$: $\quad L\dot{B}_0(t) = +S_0 e^{-\frac{t-t_0}{\tau}} \Rightarrow B_0(t) = b_0 - \frac{\tau}{L} S_0 e^{-\frac{t-t_0}{\tau}}$