Separation of variables:

$$T(x,y) = A(x)\,B(y)$$

$$\Rightarrow \quad \frac{1}{A}\frac{d^2A}{dx^2} = K$$

$$\frac{1}{B}\frac{d^2B}{dy^2} = -K$$

Note that
- if $K>0$ then $\begin{cases} A \text{ has exponential behaviour} \\ B \text{ has oscillatory behaviour} \end{cases}$
- if $K=0$ then both must be linear
- if $K<0$ then $\begin{cases} A \text{ has oscillatory behaviour} \\ B \text{ has exponential behaviour.} \end{cases}$

- looking at the boundary conditions in x ($A(0) = A(L) = 0$) we see that if A is a linear combination of $e^{\sqrt{K}x}$ and $e^{-\sqrt{K}x}$ then the only solution is $A=0$

$$\rightarrow \quad K \leq 0$$

- We can rule out $K=0$ on the same ground

$\rightarrow \quad K<0$ so define $K = -k^2$

$\Rightarrow$ for each k,

$$A_k(x) = a_k \cos kx + b_k \sin kx$$

$$B_k(y) = \alpha_k e^{ky} + \beta_k e^{-ky}$$

or equivalently

$$= \tilde{\alpha}_k \cosh(ky) + \tilde{\beta}_k \sinh(ky)$$

$$A_k(0) = A_k(L) = 0 \quad \Rightarrow \quad a_k = 0 \qquad k = \frac{n\pi}{L}$$

$$B_k(0) = 0 \quad \Rightarrow \quad \tilde{\alpha}_k = 0$$

So $T(x,y) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right)$

(note that the term for $n=0$ is 0).

To satisfy the remaining boundary condition at $y=1$ we require that

$$T(x,1) = T_0(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi}{L}\right)$$

$\Rightarrow$ This looks like a Fourier Series for an odd function periodic with period $2L$ $\Rightarrow$ let's construct the $\tilde{T}_0(x)$ extension of $T_0(x)$ with these properties, then

$$\sinh\left(\frac{n\pi}{L}\right) b_n = \frac{1}{L}\int_{-L}^{L} \tilde{T}_0(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{\sinh\left(\frac{n\pi}{L}\right)} \frac{2}{L} \int_0^L T_0(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

<u>Example</u>: Let $T_0(x) = A\sin^2\left(\frac{\pi x}{L}\right)$ then

$$\frac{2}{L}\int_0^L A\sin^2\left(\frac{\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{\cancel{2}}{L}\int_0^L A\left(\frac{1-\cos\left(\frac{2\pi x}{L}\right)}{\cancel{2}}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

Using $\cos a \sin b = \frac{1}{2}\sin(a+b) - \frac{1}{2}\sin(a-b)$

$$= \frac{A}{L}\left\{ \left[-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]_0^L - \frac{1}{2}\int_0^L \sin\left(\frac{(n+2)\pi x}{L}\right) dx + \frac{1}{2}\int_0^L \sin\left(\frac{(2-n)\pi x}{L}\right) dx \right\}$$

$$= \frac{A}{L}\left\{ \left(-\frac{L}{n\pi}\cos(n\pi) + \frac{L}{n\pi}\right) - \frac{1}{2}\frac{L}{(n+2)\pi}\left((-1)^{n+2}-1\right) - \frac{1}{2}\frac{L}{(n-2)\pi}\left((-1)^{n-2}-1\right) \right\}$$

if $n \neq 2$.

$\Rightarrow$ if n is even, $b_n = 0$.

if n is odd then

$$b_n = \frac{A}{L}\left\{ +\frac{2L}{n\pi} + \frac{L}{(n+2)\pi} + \frac{L}{(n-2)\pi} \right\} \cdot \frac{1}{\sinh\left(\frac{n\pi}{L}\right)}$$

$$= \frac{1}{\sinh\left(\frac{n\pi}{L}\right)} \left[\frac{2}{n\pi} + \frac{2n}{(4-n^2)\pi}\right] A = \frac{8A}{n(4-n^2)\pi} \frac{1}{\sinh\left(\frac{n\pi}{L}\right)}$$

So finally,

$$T(x,y) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8A}{n(4-n^2)\pi} \sin\left(\frac{n\pi x}{L}\right) \sinh\left(\frac{n\pi y}{L}\right) \frac{1}{\sinh\left(\frac{n\pi}{L}\right)}$$

$$= \sum_{p=0}^{\infty} \frac{8A \cdot \frac{1}{\sinh\left(\frac{(2p+1)\pi}{L}\right)}}{(2p+1)(4-(2p+1)^2)} \sin\left(\frac{(2p+1)\pi x}{L}\right) \sinh\left(\frac{(2p+1)\pi y}{L}\right)$$

Note: we can see that if $A=0$ ($T_0(x)=0$) then the solution in the domain is identically 0

$\Rightarrow$ This is a property of Laplace's equation: if the bcs are identically 0 on the contour then the solution is 0 everywhere.