Aside

a probabilistic derivation of the diffusion equation (Brownian motion)

Imagine a lattice (in 1D). Define the concentration $C(x,t)$ as the expected number of particles at position x, time t.

Particles have equal probability to move left or right (p) and probability to stay where they are ($1-2p$)

So $$C(x,t+\Delta t) = p\left(c(x-\Delta x,t) + c(x+\Delta x,t)\right) + (1-2p)\,c(x,t)$$

Now assume Δt small and Δx small then

$$C(x,t) + \Delta t \frac{\partial c}{\partial t} = p\left[2c(x,t) + \Delta x^2 \frac{\partial^2 c}{\partial x^2}\right] + (1-2p)c(x,t)$$
$$= c(x,t) + p\Delta x^2 \frac{\partial^2 c}{\partial x^2}$$

$$\Rightarrow \quad \frac{\partial c}{\partial t} = p\frac{\Delta x^2}{\Delta t}\frac{\partial^2 c}{\partial x^2}$$

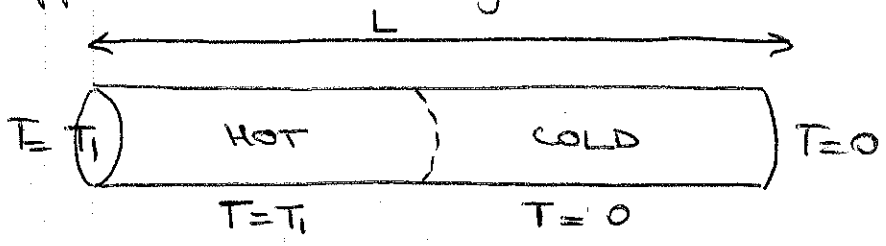
↑ define this as the diffusion coefficient k.

Note 1. In the presence of forces this derivation leads to Fokker-Planck equation.

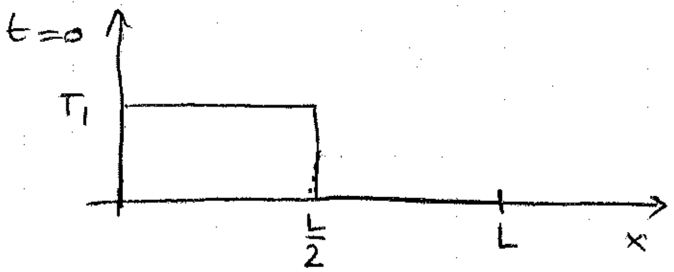
Note 2. From more general considerations, you can derive all the PDEs of fluid mechanics from statistical averaging of ensemble properties of individual particles. Kinetic theory (Boltzmann's equation and its moments).

4.2 Heat diffusion in a rod

Suppose we initially have a rod half-heated



The side walls are insulated so that heat can only be transferred laterally (x-direction).



The edges are kept at temperatures 0 and T_1 respectively.

$$\begin{cases} T(0, t) = T_1 \\ T(L, t) = 0 \end{cases}$$

The PDE is $\frac{\partial T}{\partial t} = D \frac{\partial^2 T}{\partial x^2}$

Again, we try separating the variables such that

$$T(x, t) = A(x) B(t)$$

$$\Rightarrow \quad A(x) \frac{dB}{dt} = D B(t) \frac{d^2 A}{dx^2}$$

$$\Rightarrow \quad \frac{1}{B} \frac{dB}{dt} = \frac{D}{A} \frac{d^2 A}{dx^2} = \text{constant } K$$

So
$$\begin{cases} \frac{dB}{dt} = KB \\ \frac{d^2 A}{dx^2} = \frac{KA}{D} \end{cases}$$

$\longrightarrow$ as before we expect K to be negative to satisfy the boundary conditions simultaneously, so that $K = -k^2$.

→ for each value of k there is a possible solution

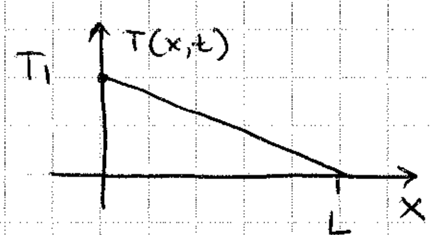
$$A_k(x) = \alpha_k \cos\left(\frac{k}{\sqrt{D}}x\right) + \beta_k \sin\left(\frac{k}{\sqrt{D}}x\right), \quad B_k(t) = e^{-k^2 t}$$

Importan Note: if $K = 0$ then there is also a solution with

$$A = ax+b \quad , \quad B = \text{constant}$$

To fit the boundary conditions, let us use our intuition about the problem

- we expect that as $t \to \infty$ the system relaxes to a temperature profile



$$T(x, t\to\infty) = T_1 - \frac{T_1}{L}x$$

→ that's the $ax+b$ part!

- The behavior of $\frac{dB}{dt} = -k^2 B$ suggests decaying exponential modes for all $k \neq 0$

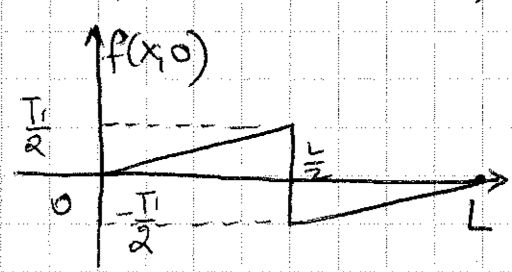
⇒ We expect the solution to be

$$T(x,t) = T_1 - \frac{T_1}{L}x + \begin{pmatrix}\text{some spatial} \\ \sin/\cos \text{ mode}\end{pmatrix} \cdot \begin{pmatrix}\text{a decaying} \\ \text{exponential}\end{pmatrix}$$

$$= T_1 - \frac{T_1}{L}x + f(x,t)$$

where $\begin{cases} f(0,t) = 0 \\ f(L,t) = 0 \end{cases}$

and $f(x,0) = T(x,0) - \left[T_1 - \frac{T_1}{L}x\right]$.



→ Now we see that if $f(0,t) = 0$ then $\alpha_k = 0$ and if $f(L,t) = 0$ then

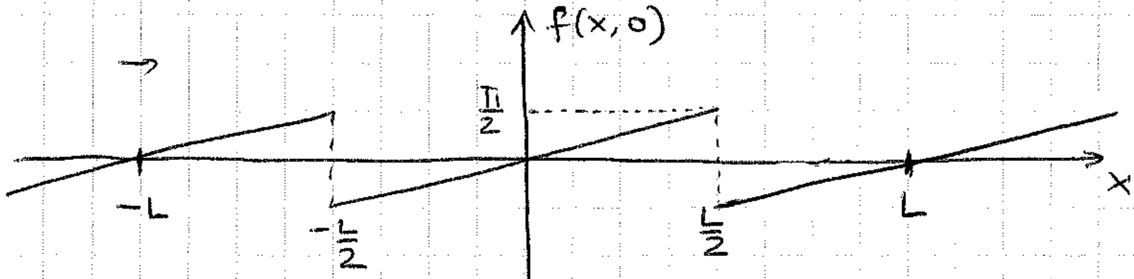
$$\frac{k}{\sqrt{D}}L = n\pi \Rightarrow k = \frac{n\pi\sqrt{D}}{L}$$

So the solution for $T(x,t)$ is

$$T(x,t) = T_1 - \frac{T_1}{L}x + \sum_1 b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2 Dt}{L^2}}$$

where $\displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x,0)$

$\Rightarrow$ Construct a Fourier Series for $f(x,0)$ by assuming
- it is periodic with period $2L$
- it is antisymmetric



So $$b_n = \frac{1}{L}\int_{-L}^{L} f(x,0)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^L f(x,0)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^{\frac{L}{2}} \frac{T_1 x}{L}\sin\left(\frac{n\pi x}{L}\right)dx$$

$$+ \frac{2}{L}\int_{\frac{L}{2}}^{L} \frac{T_1}{L}(x-L)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2T_1}{L^2}\left[x\left(\frac{-L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\right]_0^L + \frac{2T_1}{L^2}\left(+\frac{L}{n\pi}\right)\int_0^L \cos\left(\frac{n\pi x}{L}\right)dx$$

$$- \frac{2T_1}{L}\left[\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\right]_{\frac{L}{2}}^{L}$$

$$= \cancel{-\frac{2T_1}{n\pi}\cos(n\pi)} + \cancel{\frac{2T_1}{n\pi}\cos n\pi} - \frac{2T_1}{n\pi}\cos\left(\frac{n\pi}{2}\right)$$

$$\text{so} \quad b_n = -\frac{2T_1}{n\pi}\cos\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow T(x,t) = T_1\left[1 - \frac{x}{L} + \sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\right)\cos\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\pi^2 Dt}{L^2}}\right]$$

Note:

Each mode decays with a different typical timescale: the decay time for mode n is

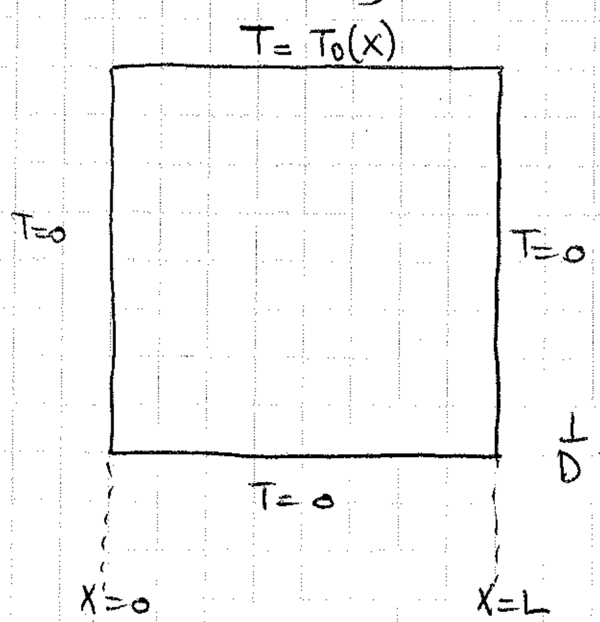
$$\tau_n = \frac{L^2}{\pi^2 n^2 D}$$

$\rightarrow$ the higher the degree n, the faster the decay

$\rightarrow$ diffusion smoothes out small scale faster than large scales.

4.3 Laplace Equation

Consider a square plate with sides held at the following temperatures:



What is the steady-state temperature profile on the plate as a result of this heating?

$\rightarrow$ Solve

$$\frac{1}{D}\frac{\partial T}{\partial t} = \boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0}$$