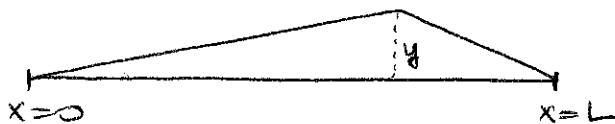


CHAPTER 4

Generic behaviour of Hyperbolic/
Parabolic/Elliptic equations through
examples. Method of separation
of variables

4.1 The wave equation: the vibrating string

Let a string be tightly stretched with ends attached
at $x=0$ and $x=L$.



Displacements away from
rest are measured by y .

The equation of motion of the string follows a
wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

with boundary condition $y(0, t) = 0$
 $y(L, t) = 0$.

Method of separation of variables

Let's assume that we can find a solution
with the form

$$y(x, t) = X(x)T(t)$$

Then $X(x) \frac{d^2 T}{dt^2} = c^2 T(t) \frac{d^2 X}{dx^2}$

$$\Leftrightarrow \frac{1}{T(t)} \frac{d^2 T}{dt^2} = \frac{c^2}{X(x)} \frac{d^2 X}{dx^2}$$

↑ a function of time only ↑ a function of x only.

This equality can only hold for all t , all x provided both sides are constant:

$$\frac{1}{T(t)} \frac{d^2 T}{dt^2} = K = \frac{c^2}{X(x)} \frac{d^2 X}{dx^2}$$

This constant is arbitrary for the moment, but it is an eigenvalue of the spatial operator $\frac{c^2 d^2}{dx^2}$

$$c^2 \frac{d^2 X}{dx^2} = K X(x)$$

⇒ Will be determined by the boundary conditions.

We want to solve
$$\begin{cases} \frac{d^2 X}{dx^2} = \frac{K}{c^2} X(x) \\ X(0) = X(L) = 0 \end{cases}$$

if $K > 0$ then the solutions are exponential
if $K < 0$ then the solutions are oscillatory.

if $K > 0$ then it is not possible to fit both bcs unless $X(x) = 0 \forall x$

so let's choose $K < 0$, and write $\frac{K}{c^2} = -k^2$

$$\Rightarrow X(x) = A \cos kx + B \sin kx$$

$$X(0) = 0 \Rightarrow A = 0$$

$$X(L) = 0 \Rightarrow \sin kL = 0 \Rightarrow k = \frac{n\pi}{L}$$

so there exist an ∞ of solutions for $X(x)$, namely

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{L}\right) \text{ with } n \in \mathbb{Z}$$

Now we must solve $\frac{d^2 T}{dt^2} = K T(t) = -c^2 k^2 T(t)$

so
$$T(t) = a \cos(ckt) + b \sin(ckt)$$

since there are an ∞ of possible values of k then for each $X_n(t)$

corresponds a $T_n(t) = a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)$

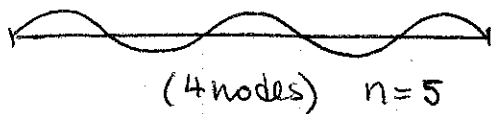
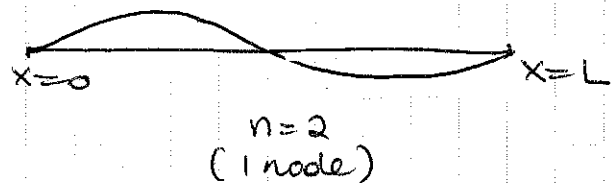
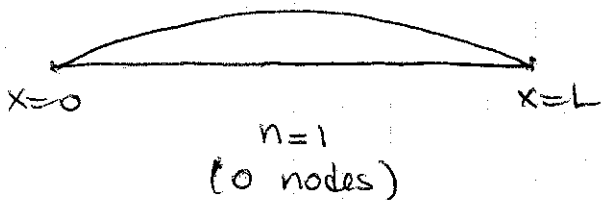
⇒ The general solution to the wave equation with these boundary conditions is a linear combination of all the solutions:

$$y(x,t) = \sum_n X_n(x) T_n(t) = \sum_n \sin\left(\frac{n\pi x}{L}\right) \left[a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right) \right]$$

(incorporate a_n with A_n and b_n with B_n + any linear combination coefficient).

Note : • This expression shows that each spatial function $\sin\left(\frac{n\pi x}{L}\right)$ vibrates with its own frequency $\frac{nc}{2L}$ intrinsic to the system.

⇒ higher c , higher frequency
 ⇒ longer L , lower frequency.

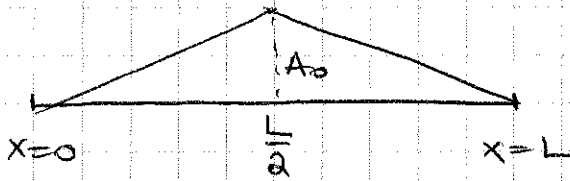


⇒ Each mode has a different frequency. The higher the degree n of the mode, the higher the frequency (the pitch of the sound emitted for example).

• To determine which mode is excited and with which amplitude it is vibrating, we need to apply initial conditions to the system.

Example: Suppose we pluck the string in the middle, so that at $t=0$ we release it from rest with

$$y(x,0) = \begin{cases} \frac{2A_0}{L}x & \text{if } 0 < x < \frac{L}{2} \\ 2A_0 - \frac{2x}{L}A_0 & \text{if } x \in [\frac{L}{2}, L] \end{cases}$$



$$\frac{\partial y}{\partial t}(x,0) = 0 \quad \text{since we release it from rest}$$

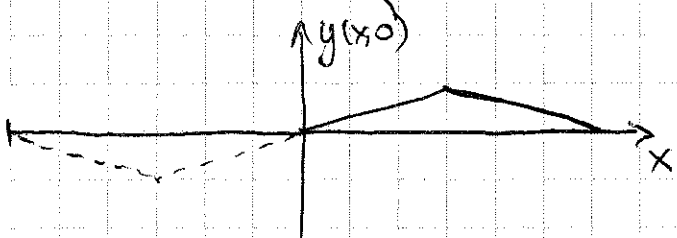
\Rightarrow then we fit the general form $y(x,t)$ to these bcs by requiring that

$$y(x,0) = \sum_n a_n \sin \frac{n\pi x}{L} = \begin{cases} \frac{2A_0 x}{L} & \text{if } x \in [0, \frac{L}{2}] \\ 2A_0 - \frac{2A_0 x}{L} & \text{if } x \in [\frac{L}{2}, L] \end{cases}$$

$$\frac{\partial y}{\partial t}(x,0) = \sum_n \frac{n\pi c}{L} b_n \sin \left(\frac{n\pi x}{L} \right) = 0 \quad \Rightarrow \boxed{b_n = 0}$$

The first of these two equations implies that we are seeking the coefficients a_n such that the series on the left is equal to the function on the RHS \Rightarrow this looks like a Fourier series problem!

Problem: $y(x,0)$ is not periodic \Rightarrow to remedy the problem, turn $y(x,0)$ into a periodic function by adding the interval $[-L, 0]$



\rightarrow construct an odd function since we are looking for a \sin expansion

$$\Rightarrow \text{By definition} \quad a_n = \frac{1}{L} \int_{-L}^L y(x,0) \sin \left(\frac{n\pi x}{L} \right) dx$$

$$\text{by symmetry} \quad = \frac{2}{L} \int_0^L y(x,0) \sin \left(\frac{n\pi x}{L} \right) dx$$

$$\begin{aligned}
 a_n &= \frac{2}{L} \left\{ \int_0^{\frac{L}{2}} \left(\frac{2A_0 x}{L} \right) \sin \frac{n\pi x}{L} dx + \int_{\frac{L}{2}}^L \left(2A_0 - \frac{2A_0 x}{L} \right) \sin \frac{n\pi x}{L} dx \right\} \\
 &= \frac{2}{L} \left\{ \left[\frac{2A_0 x}{L} \left(-\frac{L}{n\pi} \right) \cos \frac{n\pi x}{L} \right]_0^{\frac{L}{2}} + \int_0^{\frac{L}{2}} \frac{2A_0 L}{L n\pi} \cos \frac{n\pi x}{L} dx \right. \\
 &\quad + 2A_0 \left(-\frac{L}{n\pi} \right) \left[\cos \frac{n\pi x}{L} \right]_{\frac{L}{2}}^L - \left[\frac{2A_0 x}{L} \left(-\frac{L}{n\pi} \right) \cos \frac{n\pi x}{L} \right]_{\frac{L}{2}}^L \\
 &\quad \left. - \int_{\frac{L}{2}}^L \frac{2A_0 L}{L n\pi} \cos \frac{n\pi x}{L} dx \right\} \\
 &= \frac{2}{L} \left\{ \left[-\frac{2A_0 x}{n\pi} \cos \frac{n\pi x}{L} + \frac{2A_0 L}{n\pi} \sin \left(\frac{n\pi x}{L} \right) \right]_0^{\frac{L}{2}} \right. \\
 &\quad \left. + \left[-\frac{2A_0 L}{n\pi} \cos \frac{n\pi x}{L} + \frac{2A_0 x}{n\pi} \cos \frac{n\pi x}{L} - \frac{2A_0 L}{n\pi} \sin \frac{n\pi x}{L} \right]_{\frac{L}{2}}^L \right\} \\
 &= \frac{2}{L} \left\{ -\frac{2A_0 L}{n\pi} \cos \left(\frac{n\pi}{2} \right) + \frac{2A_0 L}{n^2 \pi^2} \sin \left(\frac{n\pi}{2} \right) \right. \\
 &\quad - \frac{2A_0 L}{n\pi} \cos(n\pi) + \frac{2A_0 L}{n\pi} \cos \left(\frac{n\pi}{2} \right) + \frac{2A_0 L}{n\pi} \cos(n\pi) \\
 &\quad \left. - \frac{2A_0 L}{n\pi} \cos \left(\frac{n\pi}{2} \right) - \frac{2A_0 L}{n^2 \pi^2} \sin(n\pi) + \frac{2A_0 L}{n^2 \pi^2} \sin \left(\frac{n\pi}{2} \right) \right\} \\
 &= \frac{2}{L} \left\{ \frac{4A_0 L}{n^2 \pi^2} \sin \left(\frac{n\pi}{2} \right) \right\} = \frac{8A_0}{n^2 \pi^2} \sin \left(\frac{n\pi}{2} \right) \quad |n| > 1
 \end{aligned}$$

⇒ Finally,

$$y(x,t) = \sum_{n=1}^{\infty} \frac{8A_0}{n^2 \pi^2} \sin \left(\frac{n\pi}{2} \right) \sin \left(\frac{n\pi x}{L} \right) \cos \left(\frac{n\pi c t}{L} \right)$$

amplitude
of the
mode

spatial
mode

temporal
variation of
the mode.

The solution is a superposition of all the modes vibrating independently with constant amplitude determined by their initial conditions.