II Partial Differential equations (PDEs)

① Mathematical definition

- A PDE is a functional relation between a function and its partial derivatives :

let $f : \mathbb{R}^n \to \mathbb{R}$ then the most general form of a PDE is

$$\underbrace{\mathcal{F}}_{\text{a functional of}}\left(\underbrace{x_i}_{\text{the independent variables}} ; f ; \underbrace{\frac{\partial f}{\partial x_i}}_{\text{the first order partial derivatives}} ; \underbrace{\frac{\partial^2 f}{\partial x_i \partial x_j}}_{\text{the second order partial derivatives, etc...}} ; \dots \right) = 0 \qquad \begin{cases} i = 1 \dots n \\ j = 1 \dots n \end{cases}$$

- The order of the PDE is the order of the highest derivative involved

- A PDE is said to be linear if $\mathcal{F}$ is a linear combination of f and its derivatives :

$$a_0(x_1, \dots, x_n) f + a_1(x_1 \dots x_n) \frac{\partial f}{\partial x_1} + \dots + a_n (x_1 \dots x_n) \frac{\partial f}{\partial x_n}$$
$$+ a_{11}(x_1 \dots x_n) \frac{\partial^2 f}{\partial x_1^2} + \dots \quad a_{nn}(x_1 \dots x_n) \frac{\partial^2 f}{\partial x_n^2} + \dots$$
$$= b(x_1 \dots x_n)$$

- A linear PDE is homogeneous if $b = 0$

- A non linear PDE is not linear

Examples : $f_y + f_x = 0$ is linear, homogeneous, first-order

$f_{xx} + x f_y = 0$ is linear, second order, homogeneous

$3x f_x + f_{yy} = 4$ is linear, second order, non-homogeneous

$f_y + f_x f_{yy} = 0$ is non-linear.

- There exists systems of PDEs describing the coupled evolution of two or more dependent variables

Example :
$$\begin{cases} f_x + g_y f_{xx} = 0 \\ g_{xx} = 3 f_y . \end{cases}$$
(a nonlinear coupled system of PDEs)

(2) PDEs in real systems

- Nature is classically represented by 3 spatial dimensions and one time dimension

$\Rightarrow$ Most PDEs studied involve these 4 variables, or a subset of them

- The study of PDEs is often an attempt at modelling Nature

$\Rightarrow$ There are usually 3 steps to the problem

  1. to construct a PDE which describes the problem (cf Applied Mathematics)
  2. to solve the PDE mathematically
  3. to analyse the answer in the light of the problem

- This class mostly focusses on (2) although there are two aspects of (1) worth describing here

  - notion of covariance
  - selection of a coordinate system

a. Notion of covariance

- The idea of a coordinate system is dependent on the observer / modeller. However, the physical process described by a PDE is usually independent of the observer.

  $\Rightarrow$ PDEs which describe a real phenomenon should first be written in a way that is independent of any coordinate system.

How? $\Rightarrow$ use universal differential operators:

e.g.: ∇f or $\nabla \cdot \underline{A}$ or $\nabla \times \underline{A}$

Example: The Laplace Equation: $\boxed{\nabla^2 f = 0}$

$\Rightarrow$ this is a universal equation, so that the solution to this equation, whether expressed in Cartesian coordinates (x, y, z)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

or in spherical polar coordinates (r, θ, ϕ)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2} = 0$$

represents the same scalar field f

- For a list of standard differential operators expressed in various coordinate systems see Handout.

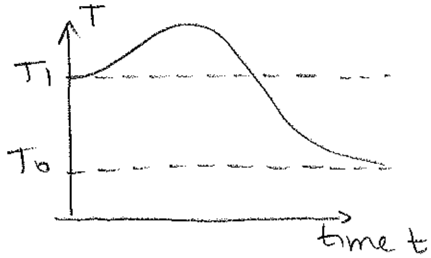
b. Selection of a coordinate system

For a given PDE, a judicious selection of a coordinate system is usually very useful. The idea is to use a system which best represents the symmetries of the physical problem

- → to model a ball : use spherical coordinates
- → to model a cylinder/cone : use cylindrical coordinates
- → to model a cube/rectangle : use Cartesian coordinates.

- Finally, it is always a good idea to use one's physical intuition of the system as a guide to find the solution or to critically assess the solution obtained.

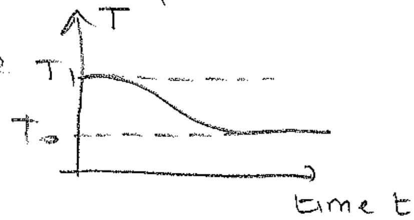
e.g. to model the cooling of a sphere initially at temperature T_1, immersed in a "bath" at temperature T_0 $\Rightarrow$ we know that the sphere is unlikely to get hotter than T_1 at any time. $\Rightarrow$ so the solution cannot be



$\Leftarrow$ NOT PHYSICAL

In fact, we also know that as $t \to \infty$, the temperature of the sphere should approach T_0

$\to$ the solution is more likely to be



- In fact, we will see that for LINEAR PDEs there exists only 3 possible types of behaviour, and these 3 are directly related to well-known physical systems

$\Rightarrow$ understanding the link between the mathematical nature of a PDE and its physical behaviour will help find solutions more easily

These are:
- the wave equation
- the heat equation (the diffusion equation)
- Laplace's equation

(3) Fundamental 2nd order PDEs, (linear)

a. The wave equation

- Generally written as

$$\frac{\partial^2 f}{\partial t^2} = c^2 \nabla^2 f$$

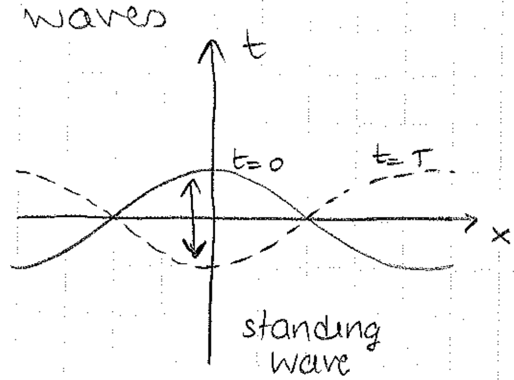
$\Rightarrow$ in cartesian coordinates for example

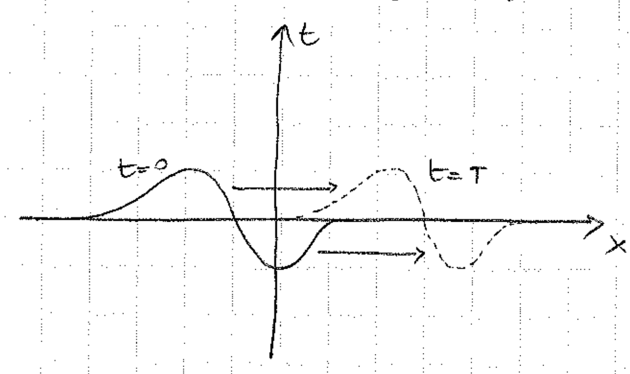
$$\frac{\partial^2 f}{\partial t^2} = c^2 \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right)$$

- The typical behaviour of the solution is <u>oscillatory</u> in time & space

- There is <u>no dissipation</u> (energy is conserved)

- There can be standing waves or propagating waves





<u>Examples in Nature</u>

- sound waves
- light (electromagnetic waves)
- seismic waves
- water waves

⋮

10. The heat equation (diffusion equation)

- Generally written as

$$\frac{\partial f}{\partial t} = \nabla \cdot (k \nabla T) = k \nabla^2 T \text{ if } k \text{ is constant}$$

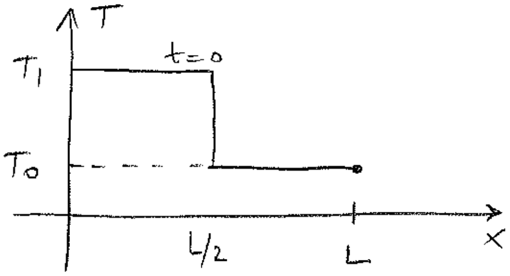
eg: in cartesian coordinates

$$\frac{\partial f}{\partial t} = k\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right)$$

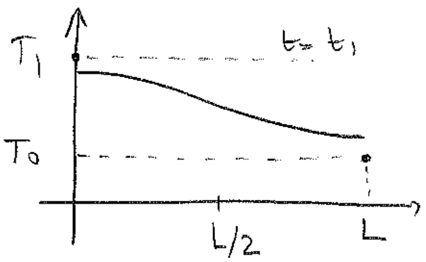
- This equation describes a diffusion process, its typical behaviour is to smooth out (and/or) dissipate gradients. The most commonly use example is the behaviour of a temperature field (hence the name "heat" equation).

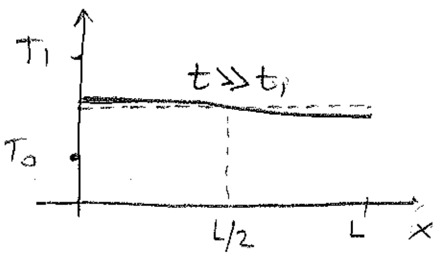
Example

If the initial condition for temperature in an insulated cylindrical thin rod of length L is



then





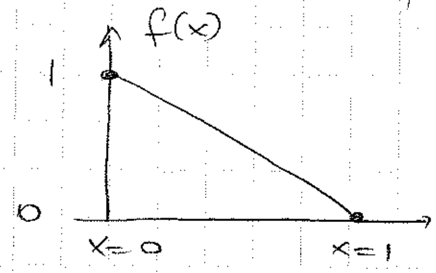
Physical intuition is very useful for diffusion problems.

c. Laplace's equation

- Generally written as $\nabla^2 f = 0$
- Note that $\nabla^2 f = 0$ is the steady-state version of the heat equation $\Rightarrow$ can be thought of as the "end-product" of a diffusion process (after waiting an ∞ time)

Example: What is the solution to $\nabla^2 f = 0$ in 1D if $\begin{cases} f(x=0) = 1 \\ f(x=1) = 0 \end{cases}$

Idea: Imagine a 1-D rod of lenght 1, with one end held at temperature 1 and the other at temperature 0. After an infinite time, the temperature in the rod has equilibrated to



→ the solution to $\nabla^2 f = 0$ with these bcs.

Check: in 1D, $\nabla^2 f = f_{xx}$ so we simply solve

$f_{xx} = 0$ with above bcs

→ solution is $f(x) = 1 - x$

(4) Fundamental 1st order PDE: the transport equation

$$\boxed{\frac{\partial f}{\partial t} + \nabla \cdot (\underline{u} f) = 0}$$

where $\underline{u}$ is a velocity field (known)

Expression in Cartesian coordinates:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial x}(u f) + \frac{\partial}{\partial y}(v f) + \frac{\partial}{\partial z}(w f) = 0$$

if

$$\underline{u} = (u, v, w)$$

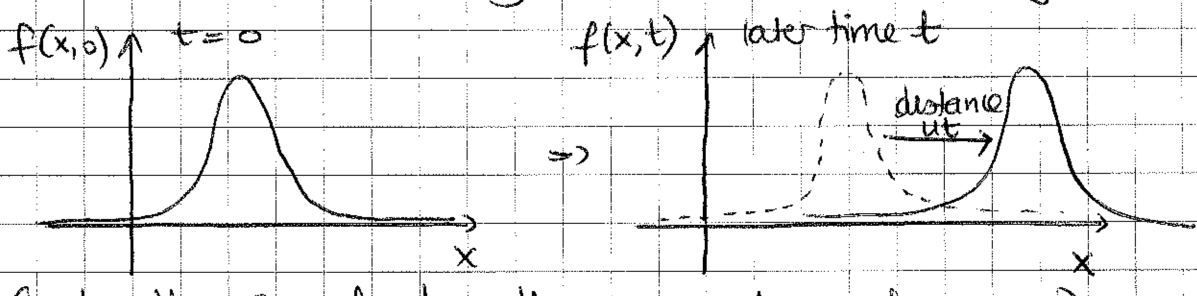
- This equation expresses the conservation of the quantity f through transport by the velocity field $\underline{u}$.

- If $\underline{u}$ is a constant field then

$$\boxed{\frac{\partial f}{\partial t} + \underline{u} \cdot \nabla f = 0}$$

(note that this is also generally true for all vector fields $\underline{u}$ s.t. $\nabla \cdot u = 0$)

A solution is merely "moved around" by $\underline{u}$.



(note the similarity with propagation of wave).

See ppt for movies in more than 1D.