Review : Elements of Fourier Series

① Periodic function

- A periodic function is a function which satisfies the relation

$$f(x) = f(x+T) \quad \text{for all } x \text{, and a given } T>0$$

T is the period of the function.

- Note that a function which is periodic with period T is also periodic with period nT for any $n \in \mathbb{N}$, $n>0$. Usually T is the smallest real value for which $f(x) = f(x+T)$ holds.

② Orthogonality

- An inner product can be defined for any two functions on an interval $[a,b]$ as

$$\langle f,g \rangle = \int_a^b f(x)g(x)w(x)\,dx$$

where $w(x)$ is a fixed positive weight function (usually satisfying $\int_a^b w(x)dx = 1$.)

- Two functions are therefore orthogonal on $[a,b]$ provided $\langle f,g \rangle = 0$.

with $w(x) = \frac{1}{2L}$

Property:
- the functions $\sin\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{m\pi x}{L}\right)$ are orthogonal on $[-L,L]$ for all (m,n)
- the functions $\sin\left(\frac{n\pi x}{L}\right)$ and $\sin\left(\frac{m\pi x}{L}\right)$ are orthogonal on $[-L,L]$ for all $m \neq n$
- the functions $\cos\left(\frac{n\pi x}{L}\right)$ and $\cos\left(\frac{m\pi x}{L}\right)$ are orthogonal on $[-L,L]$ for all $m \neq n$

(3) Fourier Series

Any function f periodic with period $2L$ can be written as the series (called a Fourier Series)

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

the Fourier coefficients

$$\begin{cases} a_0 = \dfrac{1}{2L}\displaystyle\int_{-L}^{L} f(x)\,dx \\ a_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \\ b_n = \dfrac{1}{L}\displaystyle\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx. \end{cases}$$

<u>Proof</u>: Let $m>0$

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx$$

$$= \int_{-L}^{L}\left[a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]\cos\left(\frac{m\pi x}{L}\right)dx$$

$$= \int_{-L}^{L} a_0\cos\left(\frac{m\pi x}{L}\right)dx + \int_{-L}^{L}\sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

$$+ \int_{-L}^{L}\sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx$$

given some prayer for convergence of the series

$$= \int_{-L}^{L} a_m\cos^2\left(\frac{m\pi x}{L}\right)dx = \frac{2L}{2}a_m = La_m$$

so $$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx$$ as required

(and similarly for the other terms)

④ Properties of Fourier a_n, b_n coefficients

- if $f(x)$ is an even function $\left(f(x) = f(-x)\right)$
  then $b_n = 0 \quad \forall n$

- if $f(x)$ is an odd function $\left(f(x) = -f(-x)\right)$
  then $a_n = 0 \quad \forall n$

- The Fourier coeffs. of the sum of two functions f and g is equal to the sum of the Fourier coefficients.

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$g(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)$$

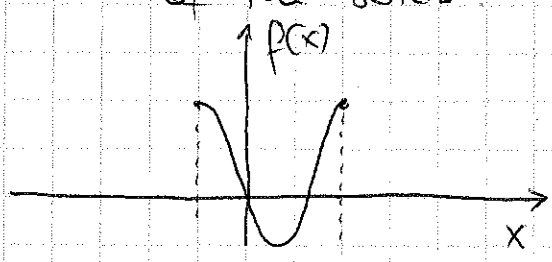
then

$$f+g\ (x) = (a_0 + A_0) + \sum_{n=1}^{\infty} (a_n + A_n) \cos\left(\frac{n\pi x}{L}\right) + (b_n + B_n) \sin\left(\frac{n\pi x}{L}\right)$$

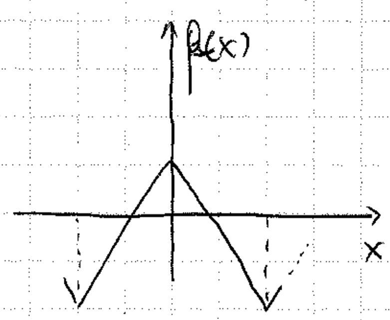
BUT : not true for the product!

- The Fourier series can be differentiated/integrated term by term to obtain the Fourier series of the derivative/integral of a function.

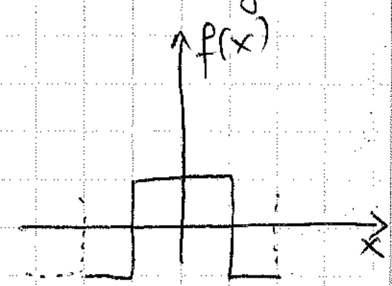
- The smoother the function, the quicker the convergence of the series:



$a_n, b_n \sim \frac{1}{n^3}$ or faster



$a_n, b_n \sim \frac{1}{n^2}$

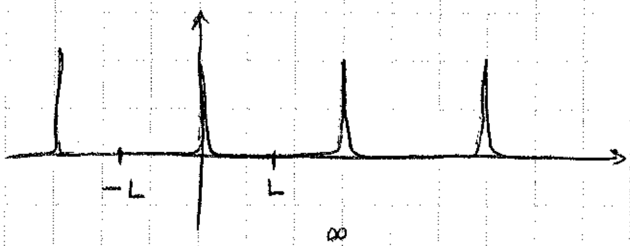


$a_n, b_n \sim \frac{1}{n}$.

Note:

- The function constructed from a Fourier Series may have different discontinuities than the one which it is trying to approximate (see HW).

- The Fourier Series for a set of δ functions is



$$f(x) = \sum_{n=0}^{\infty} \delta(x-2nL) + \delta(x+2nL)$$

so $$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{-L}^{L} \delta(x)\,dx = \frac{1}{2L}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \delta(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{\cos(0)}{L} = \frac{1}{L}$$

$$b_n = 0$$

so $$f(x) = \frac{1}{2L} + \sum_{1}^{\infty} \frac{1}{L}\cos\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{1}{2L}\sum_{-\infty}^{+\infty} \cos\left(\frac{n\pi x}{L}\right)$$