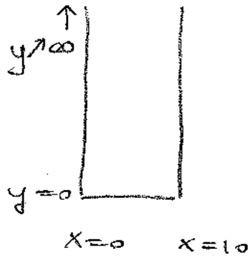
Prob 1 p548



$T(x,0) = x$
$T(0,y) = T(10,y) = 0$

$\nabla^2 T = 0$

- Separation of variables. $T(x,y) = A(x)B(y)$

$$\Rightarrow \quad \frac{A_{xx}}{A} + \frac{B_{yy}}{B} = 0 \quad \Rightarrow \quad \frac{A_{xx}}{A} = -\frac{B_{yy}}{B} = \text{constant}$$

- To fit the boundary conditions @ $x=0$ and $x=10$, we see that $A(x)$ has to be a linear combination of sin & cos function and not a linear combination of exponentials $\Rightarrow$ the constant must be negative so let

$$A_{xx} = -K^2 A \quad \rightarrow A(x) = a\cos(Kx) + b\sin(Kx)$$
$$B_{yy} = K^2 B \quad \rightarrow B(y) = \alpha e^{kx} + \beta e^{-kx}$$

- $A(0) = 0 \Rightarrow a = 0$
  $A(10) = 0 \Rightarrow K_n = \frac{n\pi}{10}$ so $A_n(x) = b_n \sin\left(\frac{n\pi x}{10}\right)$

  $B(\infty) < +\infty \Rightarrow \alpha = 0$ so $B_n(y) = e^{-\frac{n\pi}{10}y}$

- The general solution is therefore $T(x,y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right) e^{-\frac{n\pi}{10}y}$

- $T(x,0) = x = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right)$

  with $b_n = \frac{2}{10}\int_0^{10} x \sin\left(\frac{n\pi x}{10}\right)$

$$= \frac{2}{10}\left[-\frac{10}{n\pi} x \cos\left(\frac{n\pi x}{10}\right)\right]_0^{10} + \frac{2}{10}\int_0^{10} \frac{10}{n\pi}\cos\left(\frac{n\pi x}{10}\right)dx$$

$$= -\frac{20}{n\pi}\cos(n\pi) = \frac{20}{n\pi}(-1)^{n+1}$$ as required ✓

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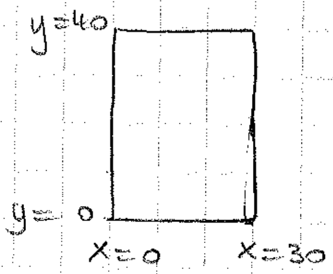

$$T(0,y) = T(30,y) = 0$$

$$T(x,0) = 0$$

$$T(x,40) = \begin{cases} 100 & x \in [0,10] \\ 0 & x \in [10,30] \end{cases}$$

- Separation of variables $T(x,y) = A(x)B(y)$

$$\Rightarrow \frac{A_{xx}}{A} + \frac{B_{yy}}{B} = 0 \Rightarrow \frac{A_{xx}}{A} = -\frac{B_{yy}}{B} = \text{constant}$$

- To fit homogeneous boundary conditions in x, we must have oscillatory functions in x $\Rightarrow$ the constant must be negative $\Rightarrow$ let

$$\begin{cases} \dfrac{A_{xx}}{A} = -k^2 & A(x) = a\cos kx + b\sin kx \\ \dfrac{B_{yy}}{B} = k^2 & B(y) = \alpha\cosh(ky) + \beta\sinh(ky) \end{cases}$$

- $A(0) = 0 \Rightarrow a = 0$
  $A(30) = 0 \Rightarrow k = \frac{n\pi}{30}$ so $A_n(x) = b_n \sin\left(\frac{n\pi x}{30}\right)$

  $B(0) = 0 \Rightarrow \alpha = 0$ so $B_n(y) = \sinh\left(\frac{n\pi y}{30}\right)$

$\Rightarrow$ General solution: $T(x,y) = \sum b_n \sin\left(\frac{n\pi x}{30}\right)\sinh\left(\frac{n\pi y}{30}\right)$

- At $y = 40$ $T(x,y) = \sum b_n \sinh\left(\frac{4n\pi}{3}\right)\sin\left(\frac{n\pi x}{30}\right) = \begin{cases} 100 & x \in [0,10] \\ 0 & x \in [10,30) \end{cases}$

$$\sinh\left(\frac{4n\pi}{3}\right) b_n = \frac{2}{30}\int_0^{30} \sin\left(\frac{n\pi x}{30}\right) T(x,40)\,dx$$

$$= \frac{2}{30}\int_0^{10} 100 \sin\left(\frac{n\pi x}{30}\right) = \frac{20}{3}\cdot\frac{30}{n\pi}\left[-\cos\frac{n\pi x}{30}\right]_0^{10}$$

$$= \frac{200}{n\pi}\left(1 - \cos\frac{n\pi}{3}\right)$$

$$\Rightarrow T(x,y) = \sum \frac{200}{n\pi}\left(1 - \cos\frac{n\pi}{3}\right)\sin\left(\frac{n\pi x}{30}\right)\frac{\sinh\left(\frac{n\pi y}{30}\right)}{\sinh(4n\pi/3)}$$

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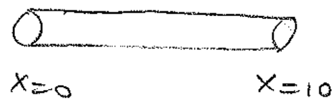

$T(x,0) = 100$
$T(0,t) = T(10,t) = 0 \quad \forall t > 0$

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$$

Separation of variables: $T(x,t) = A(x)B(t)$

$$\Rightarrow \quad k\frac{A_{xx}}{A} = \frac{B_t}{B} = \text{constant}$$

- if constant is $>0 \Rightarrow$ exponentially growing solutions, unphysical
- if constant $=0 \Rightarrow$ picks up steady-state solution solution of $A_{xx}=0$ with $A(0)=A(10)=0$
  $\rightarrow$ solution is $A_0 = 0$
- if constant is $<0 \Rightarrow$ let $\frac{1}{k}\frac{B_t}{B} = \frac{A_{xx}}{A} = -\lambda^2$

then
$$\begin{cases} A(x) = a\cos\lambda x + b\sin\lambda x \\ B(t) = e^{-\lambda^2 k t} \end{cases}$$

$A(0) = A(10) = 0 \Rightarrow a = 0$ and $\lambda_n = \frac{n\pi}{10}$

$\Rightarrow$ General solution:

$$T(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right)e^{-\frac{n^2\pi^2}{100}kt}$$

- at $t=0$ $\quad 100 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{10}\right)$

with
$$b_n = \frac{2}{10}\int_0^{10} 100\sin\left(\frac{n\pi x}{10}\right)$$
$$= 20\,\frac{10}{n\pi}\left[-\cos\frac{n\pi x}{10}\right]_0^{10}$$
$$= \frac{200}{n\pi}(1-\cos n\pi) = \frac{400}{n\pi} \text{ if } n \text{ odd.}$$

$$\Rightarrow \quad T(x,t) = \frac{400}{\pi}\sum_{\substack{\text{odd}\\ n}} \sin\left(\frac{n\pi x}{10}\right)e^{-\frac{n^2\pi^2}{100}kt}$$

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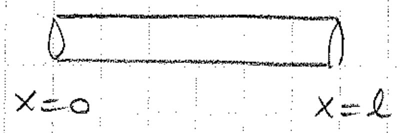

$$T(x,0) = x$$

$$\left.\frac{\partial T}{\partial x}\right|_0 = \left.\frac{\partial T}{\partial x}\right|_l = 0$$

$$\frac{\partial T}{\partial t} = k\nabla^2 T$$

- Separation of variables $T(x,t) = A(x)B(t)$
- $\Rightarrow \quad \dfrac{B_t}{B} = k\,\dfrac{A_{xx}}{A} = \text{constant}$
- The constant cannot be >0, otherwise exponentially growing solutions, unphysical
- If constant $=0$ then $A_{xx}=0$ with $\left.\frac{\partial A}{\partial x}\right|_0 = \left.\frac{\partial A}{\partial x}\right|_l = 0$
  $\Rightarrow \quad A_0(x) = a_0$
- If constant <0 then let $\dfrac{1}{k}\dfrac{B_t}{B} = \dfrac{A_{xx}}{A} = -\lambda^2$

$$\Rightarrow \begin{cases} A(x) = a\cos\lambda x + b\sin\lambda x \\ B(t) = e^{-\lambda^2 kt} \end{cases}$$

From $\left.\dfrac{dA}{dx}\right|_0 = 0 \Rightarrow b = 0$

$\left.\dfrac{dA}{dx}\right|_l = 0 \Rightarrow \lambda = \dfrac{n\pi}{l}$

$\Rightarrow$ General solution: $T(x,t) = a_0 + \displaystyle\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) e^{-n^2\pi^2 kt}$

- At $t=0$ $T(x,0) = x = a_0 + \displaystyle\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$

with $a_0 = \dfrac{1}{l}\displaystyle\int_0^l x\,dx = \dfrac{l}{2}$

$$a_n = \frac{2}{l}\int_0^l x\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{l}\left[\frac{l}{n\pi}x\sin\left(\frac{n\pi x}{l}\right)\right]_0^l - \frac{2}{n\pi}\int_0^l \sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2l^2}{n^2\pi^2}\left(\cos n\pi - 1\right) = -\frac{4l^2}{n^2\pi^2}$$ for n odd

as required