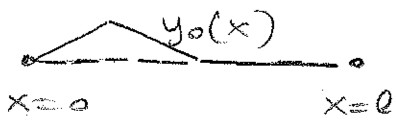
Problem 2 p 557



$$y_{tt} = c^2 y_{xx}$$
$$y(0,t) = y(\ell,t) = 0$$
$$y(x,0) = y_0(x)$$
$$y_t(x,0) = 0$$

Separation of variables : $y(x,t) = A(x)B(t)$

$$\Rightarrow \quad \frac{B_{tt}}{B} = c^2 \frac{A_{xx}}{A} = \text{constant}$$

We expect oscillatory behavior in time $\Rightarrow$ constant is negative

$\rightarrow$ let $\quad \frac{1}{c^2}\frac{B_{tt}}{B} = \frac{A_{xx}}{A} = -\lambda^2$

$$\Rightarrow \quad \begin{cases} A(x) = a\cos\lambda x + b\sin\lambda x \\ B(t) = \alpha\cos c\lambda t + \beta\sin c\lambda t \end{cases}$$

$A(0) = A(\ell) = 0 \Rightarrow a = 0$ and $\lambda = \frac{n\pi}{\ell}$

$\Rightarrow$ General solution is

$$y(x,t) = \sum_{n=1} \sin\left(\frac{n\pi x}{\ell}\right)\left[\alpha_n \cos\left(\frac{n\pi ct}{\ell}\right) + \beta_n \sin\left(\frac{n\pi ct}{\ell}\right)\right]$$

- $y_t(x,0) = 0 \Rightarrow \beta_n = 0 \quad \forall n$
- $y(x,0) = y_0(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{\ell}\right)\alpha_n$

$\Longrightarrow$

$$\alpha_n = \frac{2}{\ell}\int_0^{\ell} y_0(x)\sin\left(\frac{n\pi x}{\ell}\right)dx = \frac{2}{\ell}\int_0^{\frac{\ell}{2}} y_0(x)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$= \frac{2}{\ell}\int_0^{\ell/4} \frac{hx}{\frac{\ell}{4}}\sin\left(\frac{n\pi x}{\ell}\right)dx$$

$$+ \frac{2}{\ell}\int_{\ell/4}^{\ell/2}\left(2h - \frac{hx}{\frac{\ell}{4}}\right)\sin\left(\frac{n\pi x}{\ell}\right)dx$$

But $\displaystyle\int_a^b x\sin\lambda x\,dx = \frac{1}{\lambda}\left[a\cos\lambda a - b\cos\lambda b\right]$

$$+ \frac{1}{\lambda^2}\left(\sin\lambda b - \sin\lambda a\right)$$

So $\alpha_n = \frac{8}{\ell^2} h \left\{ \left[\frac{\ell}{n\pi}\right]\left[-\frac{\ell}{4}\cos\left(\frac{n\pi}{\cancel{\ell}}\cdot\frac{\cancel{\ell}}{4}\right)\right] + \frac{\ell^2}{n^2\pi^2}\sin\left(\frac{n\pi}{\ell}\frac{\ell}{4}\right)\right\}$

$+ \frac{4h}{\ell}\frac{\ell}{n\pi}\left[\cos\frac{n\pi}{\cancel{\ell}}\frac{\cancel{\ell}}{4} - \cos\frac{n\pi}{\cancel{\ell}}\frac{\cancel{\ell}}{2}\right]$

$- \frac{8h}{\ell^2}\left\{\frac{\ell}{n\pi}\left[\cancel{\frac{\ell}{4}\cos\frac{n\pi}{\ell}\frac{\ell}{4}} - \cancel{\frac{\ell}{2}\cos\frac{n\pi}{2}\frac{\ell}{2}}\right]\right.$

$\left. + \frac{\ell^2}{n^2\pi^2}\left(\sin\frac{n\pi}{\ell}\frac{\ell}{2} - \sin\frac{n\pi}{\ell}\frac{\ell}{4}\right)\right\}$

$= \frac{16h}{n^2\pi^2}\sin\left(\frac{n\pi}{4}\right) - \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$

as required

$\boxed{\text{Problem 5 p 558}}$

String with $y_{tt} = c^2 y_{xx}$

$y(0,t) = y(\ell,t) = 0$

$y(x,0) = 0$

$y_t(x,0) = \begin{cases} \frac{hx}{\ell/2} & \text{if } x \in [0, \ell/2] \\ 2h - \frac{hx}{\ell/2} & \text{if } x \in [\ell/2, \ell] \end{cases}$

The start is the same as the previous problem

$\Rightarrow$ General solution is

$y(x,t) = \sum \sin\left(\frac{n\pi x}{\ell}\right)\left(\alpha_n \cos\left(\frac{n\pi ct}{\ell}\right) + \beta_n \sin\left(\frac{n\pi ct}{\ell}\right)\right)$

- $y(x,0) = 0 \Rightarrow \boxed{\alpha_n = 0}$

$y_t(x,0) = f(x) \Leftrightarrow \sum \beta_n \frac{n\pi c}{\ell}\sin\left(\frac{n\pi x}{\ell}\right) = f(x)$

so $\frac{n\pi c}{l}\beta_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

$$= \frac{2}{l}\int_0^{l/2}\frac{hx}{l/2}\sin\left(\frac{n\pi x}{l}\right)dx + \frac{2}{l}\int_{l/2}^{l}\left(2h - \frac{hx}{l/2}\right)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{4h}{l^2}\left\{\frac{l}{n\pi}\left(\cancel{-\frac{l}{2}\cos\frac{n\pi}{l}\cdot\frac{l}{2}}\right) + \frac{l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{l}\frac{l}{2}\right)\right\}$$

$$+ \frac{2}{l}\cdot 2h\cdot\left[\frac{l}{n\pi}\right]\left(\cancel{\cos\frac{n\pi}{l}\cdot\frac{l}{2}} - \cancel{\cos\frac{n\pi}{l}l}\right)$$

$$- \frac{4h}{l^2}\left\{\frac{l}{n\pi}\left(\cancel{\frac{l}{2}\cos\left(\frac{n\pi}{l}\frac{l}{2}\right)} - \cancel{l\cos\left(\frac{n\pi}{l}l\right)}\right) + \frac{l^2}{n^2\pi^2}\left(\cancel{\sin\left(\frac{n\pi}{l}l\right)} - \sin\left(\frac{n\pi}{l}\frac{l}{2}\right)\right)\right\}$$

$$= \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$\Rightarrow \quad \beta_n = \frac{8hl}{n^3\pi^3 c}\sin\left(\frac{n\pi}{2}\right)$ as required