Homework 3

Problem 1

$$\begin{cases} x u u_x + y u u_y = u^2 - 1 \\ u(x, x^2) = x^3 \quad \text{for } x > 0 \end{cases}$$

let $v = u^2 - 1$ then

$$\begin{matrix} v_x = u u_x \\ v_y = u u_y \end{matrix} \Rightarrow \frac{x}{2} v_x + \frac{y}{2} v_y = v$$

↑ a linear equation!

The initial conditions become

$$v(x, x^2) = x^6 - 1$$

$$\Rightarrow \begin{cases} x_0(s) = s \\ y_0(s) = s^2 \\ v_0(s) = s^6 - 1 \end{cases}$$

Transversality condition:

$$\begin{vmatrix} 1 & \frac{s}{2} \\ 2s & \frac{s^2}{2} \end{vmatrix} = \frac{s^2}{2} - s^2 = -\frac{s^2}{2}$$

↳ problem at $x=0$ or $y=0$.

Characteristic equations

$$\begin{cases} \frac{dx}{d\tau} = \frac{x}{2} \Rightarrow x = s e^{\tau/2} \\ \frac{dy}{d\tau} = y/2 \Rightarrow y = s^2 e^{\tau/2} \\ \frac{dv}{d\tau} = v \Rightarrow v = (s^6 - 1) e^{\tau} \end{cases}$$

So $v(s, \tau) = (s^6 - 1) e^{\tau}$

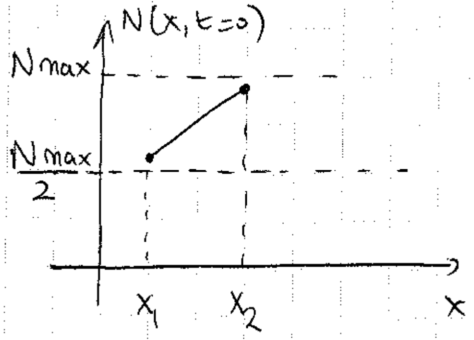
$$s = \frac{y}{x} \qquad \tau = 2\ln\left(\frac{x}{s}\right) \qquad e^{\tau} = \frac{x^2}{s^2} = \frac{x^4}{y^2}$$

$$\Rightarrow \quad v(x,y) = \left(\left(\frac{y}{x}\right)^6 - 1 \right) \frac{x^4}{y^2}$$

$$\Rightarrow \quad u(x,y) = \sqrt{v+1} = \sqrt{\frac{x^4}{y^2}\left(\frac{y^6}{x^6} - 1\right) + 1} = \sqrt{\frac{y^4}{x^2} - \frac{x^4}{y^2} + 1}$$

Problem 2

(a)



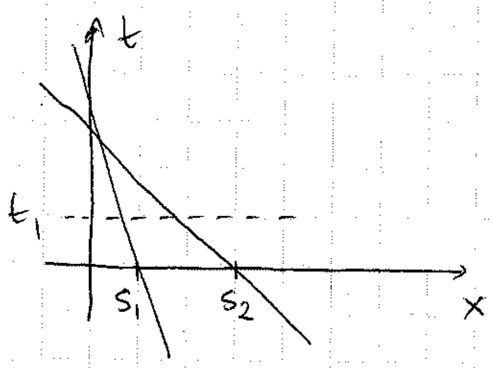
$$N(x_1, t=0) < N(x_2, t=0)$$

$$\Rightarrow \phi(s_1) < \phi(s_2)$$

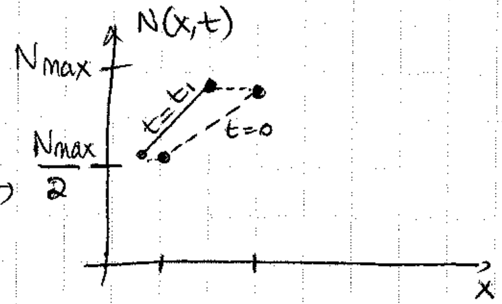
$$\Rightarrow F'(\phi(s_1)) > F'(\phi(s_2))$$

$$\Rightarrow \frac{1}{F'(\phi(s_1))} < \frac{1}{F'(\phi(s_2))}$$

(but both now negative).

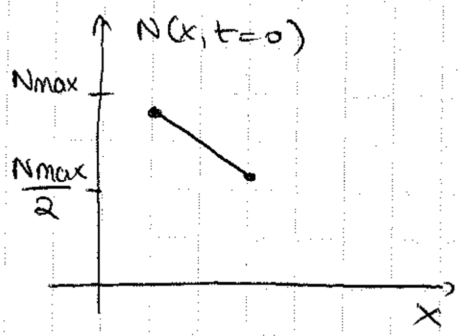


$\Rightarrow$

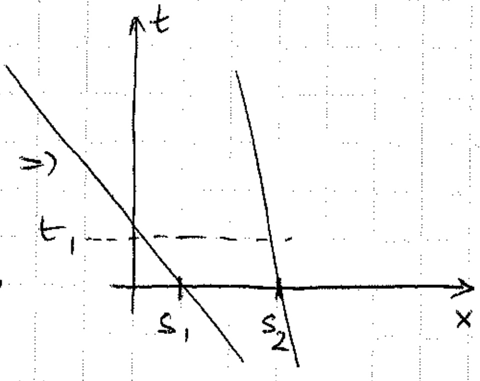


→ the front steepens & moves backward

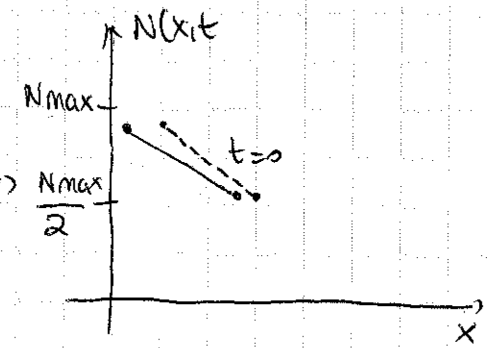
(b) By contrast



$\Rightarrow$



$\Rightarrow$



→ The front moves backward but becomes shallower.

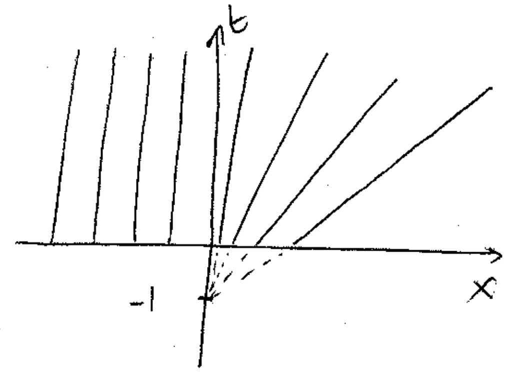
Problem 3

$$\begin{cases} u_t + u u_x = 0 \\ u(x,0) = \begin{array}{ll} x & \text{for } x>0 \\ 0 & \text{for } x \le 0 \end{array} \end{cases}$$

a) Characteristics:

$$t = \frac{x-s}{F'(\phi(s))} \quad \text{with} \quad F'(u) = u$$

$$\phi(s) = \begin{cases} s & \text{if } s>0 \\ 0 & \text{if } s<0 \end{cases}$$



$$\rightarrow \begin{cases} t = \dfrac{x-s}{s} & \text{if } s>0 \\ x = s & \text{if } s<0 \end{cases}$$

b) Solution

$$u = \phi(s) = \begin{cases} s & \text{if } s>0 \\ 0 & \text{otherwise} \end{cases}$$

but if $s>0$, $s = \dfrac{x}{t+1}$ so $u(x,t) = \begin{cases} 0 & \text{if } x<0 \\ \dfrac{x}{t+1} & \text{if } x \ge 0 \end{cases}$

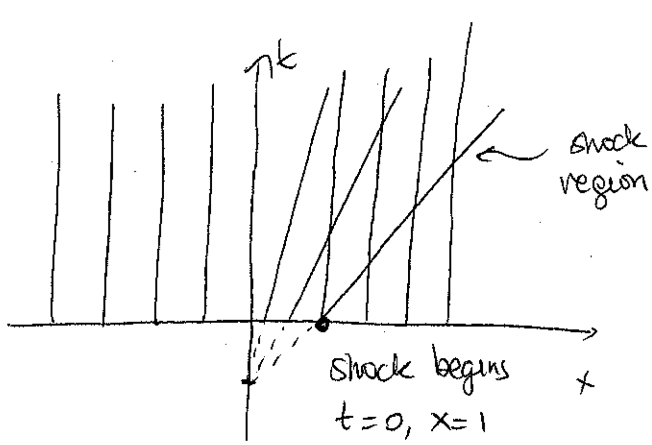
(c) Domain of definition: $t>1$, all x

Function is continuous for $t>1$, all x

Function is not differentiable at $x=0$

(d) if $\begin{cases} u(x,0) = & 0 & x<0 \\ & x & x\in[0,1] \\ & 0 & x>1 \end{cases}$

then



$$\frac{d\gamma}{dt} = \frac{F(u^+) - F(u^-)}{u^+ - u^-} \qquad F(u) = \frac{u^2}{2}$$

with $u^- = \dfrac{x}{t+1}$ $\quad u^+ = 0$.

$$\Rightarrow \quad \frac{d\gamma}{dt} = \frac{0 - \frac{1}{2}\left(\frac{\gamma}{t+1}\right)^2}{0 - \frac{\gamma}{t+1}} = \frac{1}{2}\,\frac{\gamma}{t+1}$$

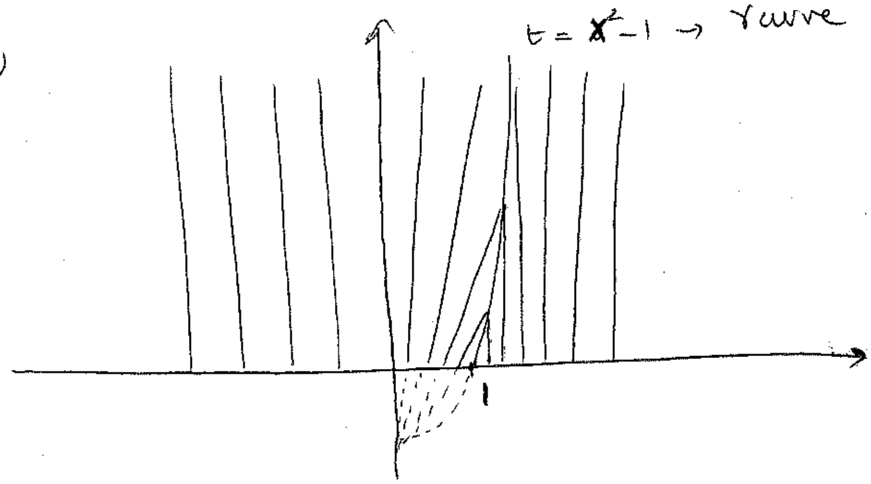
So $\quad \dfrac{d\gamma}{\gamma} = \dfrac{1}{2}\,\dfrac{dt}{t+1} \quad \rightarrow \quad \ln \gamma = \frac{1}{2}\ln(t+1)$

$$\gamma = K\sqrt{t+1}$$

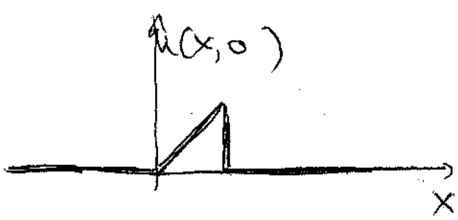
at $t=0$ $\quad \gamma = 1 \Rightarrow \quad K = 1 \quad$ so $\gamma(t) = \sqrt{t+1}$

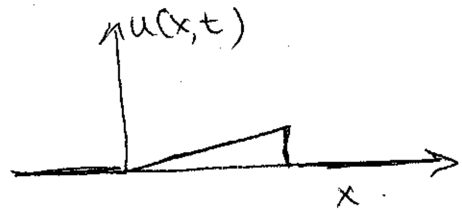
(e)

$t = x^2 - 1 \rightarrow \gamma$ curve



$$u(x,t) = \begin{cases} 0 & \text{for } x < 0 \\ \dfrac{x}{t+1} & \text{for } x \in [0, \gamma(t) = \sqrt{t+1}\,] \\ 0 & \text{for } x > \gamma(t). \end{cases}$$





Problem 4

$$\begin{cases} u_t + t u u_x = 0 \\ u(x,0) = \sin x \quad \text{for all } x. \end{cases}$$

- characteristic equations:

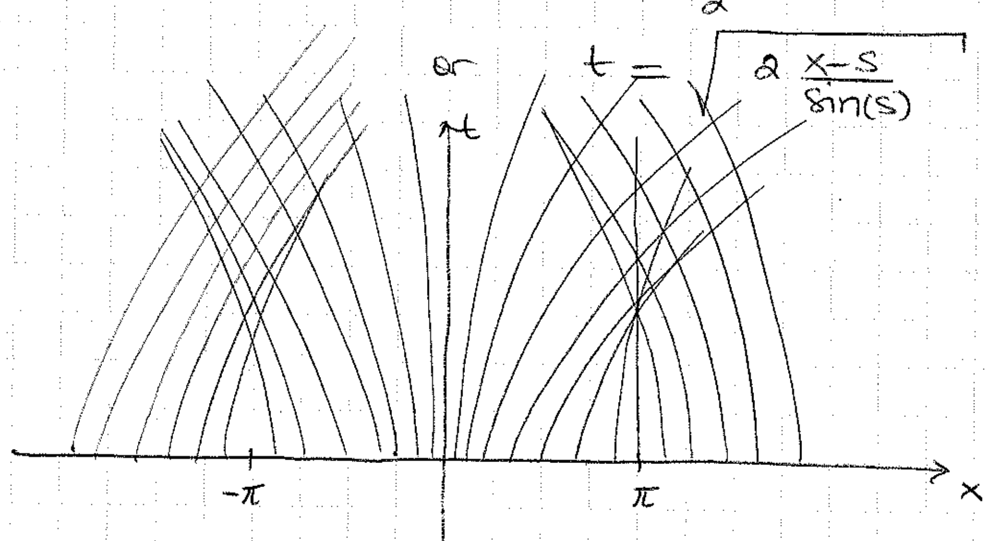
$\frac{dt}{dz} = 1 \qquad t = z$

$\frac{dx}{dz} = tu \qquad \frac{dx}{dz} = z\sin(s) \quad \text{so} \quad x = \frac{z^2}{2}\sin(s) + s$

$\frac{du}{dz} = 0 \quad \Rightarrow \quad u = \sin(s)$

The characteristics are $x = \frac{t^2}{2}\sin(s) + s$

or $t = \sqrt{2\frac{x-s}{\sin(s)}}$



- Two characteristics from s_1 and s_2 intersect at

$$\frac{t_+^2}{2}\sin(s_1) + s_1 = \frac{t_+^2}{2}\sin(s_2) + s_2$$

$$\Rightarrow \quad t_+^2 = 2\,\frac{s_2 - s_1}{\sin(s_1) - \sin(s_2)}$$

$t_+(s_1, s_2)$ is minimal when $\frac{\partial t_+}{\partial s_1} = \frac{\partial t_+}{\partial s_2} = 0$

$$\Rightarrow \begin{cases} \dfrac{-1}{\sin(s_1)-\sin(s_2)} - \dfrac{(s_2-s_1)}{[\sin(s_1)-\sin(s_2)]^2} \cdot \cos(s_1) = 0 \\ \dfrac{1}{\sin(s_1)-\sin(s_2)} - \dfrac{(s_2-s_1)}{[\sin(s_1)-\sin(s_2)]^2} \cdot (-\cos(s_2)) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} -[\sin(s_1) - \sin(s_2)] - (s_2-s_1)\cos(s_1) = 0 \\ \sin(s_1)-\sin(s_2) + (s_2-s_1)\cos(s_2) = 0. \end{cases}$$

$\rightarrow$ this is clearly true when $s_1 = s_2$

let $s_2 = s_1 + \varepsilon$ then

$$t_+^2 = 2 \frac{\varepsilon}{\sin(s_1) - \sin(s_1+\varepsilon)}$$

$$= 2 \frac{\varepsilon}{\sin(s_1) - [\sin(s_1)\cos(\varepsilon) + \cos(s_1)\sin(\varepsilon)]}$$

$$\simeq \frac{-2}{\cos(s_1)}$$

this quantity is minimized with $t_+^2 > 0$ when $\cos(s_1) = -1$

$$\Rightarrow \boxed{s_1 = s_2 = \pm\pi} \quad (+2k\pi)$$

$$\Rightarrow \quad t_c = \sqrt{2} \quad \text{and} \quad x_c = \pi + 2k\pi.$$

- For $t < t_c$, then we have

$$u = \sin(s) = \sin\left(x - \frac{t^2}{2}u\right)$$

- Change of variable: let $T = t^2$ then

$u_t = 2t\,u_T \Rightarrow$ the equation becomes

$$\boxed{u_T + \frac{u}{2}u_x = 0}$$

or, in conservative form:

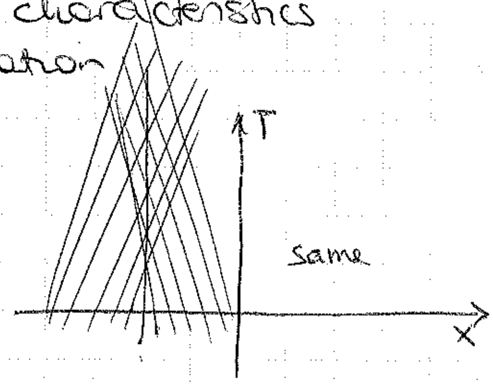
$$u_T + \frac{\partial}{\partial x}\left(\frac{u^2}{4}\right) = 0$$

- In the new coordinate system, characteristics are straight lines with equation

$$X = \frac{\sin(s)}{2} T + s$$

or

$$T = \frac{2(X-s)}{\sin(s)}$$



The shock jump condition is

$$\frac{d\gamma}{dt} = \frac{F(u_+) - F(u_-)}{u_+ - u_-}$$

by symmetry, $u_+ = -u_-$

since $\sin(-\pi + x) = -\sin(-\pi - x)$ for all x

$\sin(\pi + x) = -\sin(\pi - x)$

but $F(u) = \frac{u^2}{4}$ so $F(u_+) = F(u_-)$

$$\Rightarrow \frac{d\gamma}{dt} = 0.$$

- So the solution $u = \sin(s)$ is valid at all times

Problem 3

$$\begin{cases} u_t + c_1 u_x + c_2 u_y + \alpha u = 0 \\ u(x,y,0) = u_0(x,y) \end{cases}$$

Parametrize initial conditions as

$$\begin{cases} t_0(s_1,s_2) = 0 \\ x_0(s_1,s_2) = s_1 \\ y_0(s_1,s_2) = s_2 \\ u_0(s_1,s_2) = u_0(s_1,s_2) \end{cases}$$

Characteristic equations:

$$\frac{dt}{d\tau} = 1 \quad \Rightarrow \quad t = \tau$$

$$\frac{dx}{d\tau} = c_1 \quad \Rightarrow \quad x = c_1\tau + s_1$$

$$\frac{dy}{d\tau} = c_2 \quad \Rightarrow \quad y = c_2\tau + s_2$$

$$\frac{du}{d\tau} = -\alpha u \quad \Rightarrow \quad u = u_0(s_1,s_2)e^{-\alpha\tau}$$

So the solution is

$$\boxed{u = u_0(x - c_1 t,\ y - c_2\tau)e^{-\alpha t}}$$