HOMEWORK 2

- $u_t + V_0 u_x = 0$

  $u(x,0) = e^{-x^2/2}$

$$\rightarrow \begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = e^{-x^2/2} \end{cases} \qquad \begin{cases} \frac{dx}{d\tau} = V_0 & \rightarrow x(\tau) = V_0 \tau + s \\ \frac{dt}{d\tau} = 1 & \rightarrow t(\tau) = \tau \\ \frac{du}{d\tau} = 0 & \rightarrow u(\tau) = e^{-s^2/2} \end{cases}$$

so

$$t = \tau$$

$$s = x - V_0 t$$

$$u = e^{-(x-V_0t)^2/2} \rightarrow \text{a travelling Gaussian}$$

$$\int_{-\infty}^{+\infty} u(x,t)\,dx = \int_{-\infty}^{+\infty} e^{-(x-V_0t)^2/2}\,dx = \sqrt{2\pi}$$

Physically: This integral is constant because the solution is advected at constant velocity without change of shape.

Mathematically: To see why it is constant note that

$$\int_{-\infty}^{+\infty} u_t + V_0 u_x = 0$$

$$\rightarrow \frac{\partial}{\partial t}\int_{-\infty}^{+\infty} u\,dx + V_0 \int_{-\infty}^{+\infty} u_x\,dx = 0$$

$$\rightarrow \frac{\partial}{\partial t}\int_{-\infty}^{+\infty} u\,dx + V_0 \underbrace{\left(\text{"}u(+\infty) - u(-\infty)\text{"}\right)}_{\text{both are } 0} = 0$$

$$\rightarrow \frac{\partial}{\partial t}\int_{-\infty}^{+\infty} u\,dx = 0$$

$$\begin{cases} u_t + x u_x = -u \\ u(x,0) = e^{-x^2/2} \end{cases}$$

① $$\begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = e^{-s^2/2} \end{cases}$$

② $$\begin{cases} \dfrac{dt}{d\tau} = 1 & \to t = \tau \\ \dfrac{dx}{d\tau} = x & \to x = s e^{\tau} \\ \dfrac{du}{d\tau} = -u & \to u = u_0(s) e^{-\tau} = e^{-\frac{s^2}{2} - \tau} \end{cases}$$

Since $\tau = t$, $s = x e^{-\tau} = x e^{-t}$

we have

$$u(x,t) = e^{-t} e^{-\frac{(xe^{-t})^2}{2}} = e^{-t} e^{-\frac{1}{2}x^2 e^{-2t}}$$

→ a Gaussian with decaying amplitude and expanding width.

Note that we still have $\int u(x,t) =$ constant.

Indeed this time

$$\int_{-\infty}^{+\infty} u(x,t)\,dx = e^{-t} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\frac{x^2}{e^{2t}}} dx = e^{-t} \cdot \sqrt{2\pi e^{2t}} = \sqrt{2\pi}$$

<u>Why?</u> Note that

$$u_t + x u_x = -u \iff u_t + (u x)_x = 0$$

So if we define $v(x,t) = ux$

$v_t + v_x = 0$ → a transport equation for $v(x,t)$ with velocity 1.

So the integral of v is constant with time, and so the integral of u also

(swept under the carpet is the fact that $\int v$ is not defined)

$$\begin{cases} u_t + x u_x = e^u \\ u(x,0) = x^2 \end{cases}$$

① $\begin{cases} x_0(s) = s \\ t_0(s) = 0 \\ u_0(s) = s^2 \end{cases}$

② $\begin{cases} \frac{dt}{d\tau} = 0 & \to \quad t = \tau \\ \frac{dx}{d\tau} = x & \to \quad x = s e^{\tau} \end{cases}$

$\frac{du}{d\tau} = e^u \quad \to \quad \frac{du}{e^u} = d\tau$

$\to \quad e^{-u} du = d\tau$

$\to \quad -e^{-u} = \tau + K$

Plugging IC $\Rightarrow \quad K = -e^{-s^2}$

So $-e^{-u} = \tau - e^{-s^2}$

$-u = \ln\left(e^{-s^2} - \tau\right)$

$u = -\ln\left(e^{-s^2} - \tau\right)$

now $s = x e^{-\tau} = x e^{-t}$

and so

$$u(x,t) = -\ln\left(e^{-x^2 e^{-2t}} - t\right)$$

Note: this solution is not always defined, has a finite-time blowup.

$\boxed{\text{EXERCISES p 58-59}}$

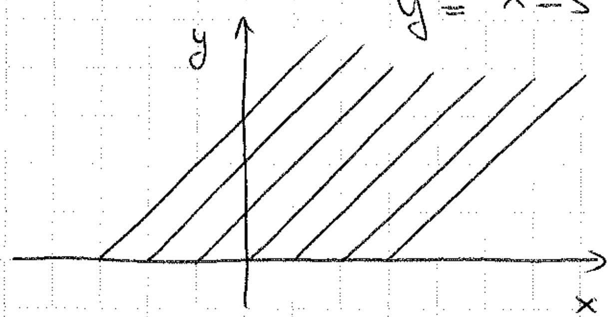
$\boxed{2.1}$ $u_x + u_y = 1$ $\quad u(x,0) = f(x) \quad \Rightarrow \begin{cases} x_0(s) = s \\ y_0(s) = 0 \\ u_0(s) = f(s) \end{cases}$

The characteristic equations are

$$\begin{cases} \dfrac{dx}{d\tau} = 1 & \Rightarrow x = \tau + s \\ \dfrac{dy}{d\tau} = 1 & \Rightarrow y = \tau \\ \dfrac{du}{d\tau} = 1 & \Rightarrow u = \tau + f(s) \end{cases}$$

so the projection of the characteristic curves in $(x-y)$ plane have equation

$y = x - s$ $\rightarrow$ straight lines, slope 1, x-intercept s



Solution: $\begin{cases} u = \tau + f(s) \\ x = \tau + s \\ y = \tau \end{cases} \Rightarrow \begin{cases} \tau = y \\ s = x - y \\ \boxed{u = y + f(x-y)} \end{cases}$

$\boxed{2.2}$ $x u_x + (x+y) u_y = 1$ $\quad u(1,y) = y \Rightarrow \begin{cases} x_0(s) = 1 \\ y_0(s) = s \\ u_0(s) = s \end{cases}$

<u>Characteristic equations</u> $\begin{cases} \dfrac{\partial x}{\partial \tau} = x \\ \dfrac{\partial y}{\partial \tau} = x + y \\ \dfrac{\partial u}{\partial \tau} = 1 \end{cases}$

$$\Rightarrow \begin{cases} x = e^{\tau} \\ \dfrac{dy}{d\tau} - y = e^{\tau} \quad \Rightarrow \dfrac{\partial}{\partial \tau}\left(y e^{-\tau}\right) = 1 \Rightarrow y e^{-\tau} - s = \tau \\ u = \tau + s \end{cases}$$

So $\begin{cases} x = e^{\tau} \\ y = e^{\tau}(\tau+s) \\ u = \tau+s \end{cases}$ $\quad \frac{y}{x} = \tau+s$ so $\boxed{u = \frac{y}{x}}$

$\Rightarrow$ the solution is not defined for $x=0$.

$\boxed{2.3}$ $\quad xu_x + yu_y = pu \quad -\infty < x < +\infty \quad -\infty < y < +\infty$

(a) Characteristic equations

$$\frac{\partial u}{\partial \tau} = pu \Rightarrow u = u_0(s)e^{p\tau}$$
$$\frac{\partial x}{\partial \tau} = x \Rightarrow x = x_0(s)e^{\tau}$$
$$\frac{\partial y}{\partial \tau} = y \Rightarrow y = y_0(s)e^{\tau}$$

(b) $p=4$, initial curve is $x^2+y^2=1$ with $u=1$

. first we parameterize it as $\begin{cases} x_0(s) = \cos(s) \\ y_0(s) = \sin(s) \\ u_0(s) = 1 \end{cases}$

then $\begin{cases} u = e^{4\tau} \\ x = \cos(s)e^{\tau} \\ y = \sin(s)e^{\tau} \end{cases}$

$\Rightarrow \quad x^2+y^2 = e^{2\tau} \Rightarrow \boxed{u = (x^2+y^2)^2}$

(c) $p=2$, initial curve is $u(x,0) = x^2$ for $x > 0$

$\begin{cases} x_0(s) = s \\ y_0(s) = 0 \\ u_0(s) = s^2 \end{cases}$

$\Rightarrow \begin{cases} u = s^2e^{2\tau} \\ x = se^{\tau} \\ y = 0 \end{cases} \Rightarrow$ A possible solution $u = x^2$

But also any function of the kind $x^2 + cy^2$ works where $c \in \mathbb{R}$:

$$x \cdot u_x + y u_y = 2x^2 + 2cy^2 = 2(x^2 + cy^2)$$

(d) Why doesn't it violate the existence & uniqueness theorem?

Transversality condition:

$$\begin{pmatrix} dx_0/ds \\ dy_0/ds \\ 0 \end{pmatrix} \times \begin{pmatrix} x_0(s) \\ y_0(s) \\ 0 \end{pmatrix}$$

$$= \frac{dx_0}{ds} y_0(s) - \frac{dy_0}{ds} x_0(s) = 1 \cdot 0 - 0 \cdot s = 0 \text{ everywhere}$$

So in fact the transversality condition is violated everywhere → theorem doesn't hold here, and there can be an ∞ of solutions.

$\boxed{2.4}$ $\begin{cases} y u_x - x u_y = 0 & (y > 0) \\ \text{(a), (b) or (c)} \end{cases}$

- characteristics:

$$\begin{cases} \dfrac{dx}{d\tau} = y \\ \dfrac{dy}{d\tau} = -x \end{cases} \quad \text{with} \begin{cases} x_0(s) = s \\ y_0(s) = 0 \end{cases}$$

$$\hookrightarrow \frac{d^2x}{d\tau^2} = -x \quad \longrightarrow \quad x(\tau) = A\cos\tau + B\sin\tau$$

$$y(\tau) = -A\sin\tau + B\cos\tau$$

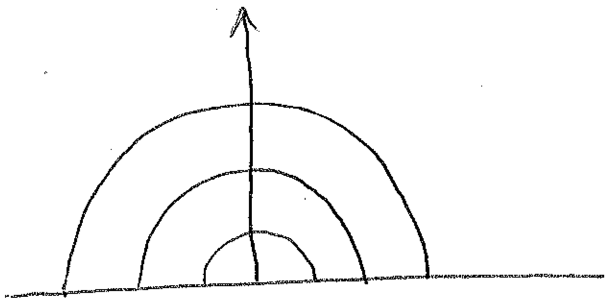
Applying ICs yields $A = s \quad B = 0$

$$\begin{cases} x(\tau) = s\cos\tau \\ y(\tau) = -s\sin\tau \end{cases}$$

So characteristics are circles $x^2 + y^2 = s^2$

$\rightarrow$ circles of radius $|s|$

<u>Case</u> (a): $\frac{du}{d\tau} = 0 \Rightarrow u(\tau) = u_0(s) = s$



u is constant on characteristics, it's value is s.

<u>Problem!</u> what value does u take?

$u(x,y) = +\sqrt{x^2+y^2}$

or $u(x,y) = -\sqrt{x^2+y^2}$?

The characteristic $s=1$ is the <u>same curve</u> as the characteristic $s=-1$

$\rightarrow$ (a) has no solution

<u>Case (b)</u> $u(\tau) = u_0(s) = s^2$

$\rightarrow$ this works : $u(x,y) = x^2+y^2$.

Solution is defined everywhere

<u>Case (c)</u> $u(\tau) = u_0(s) = s$ for $s>0$ only

$\rightarrow$ then it works, we know to take $u(x,y) = +\sqrt{x^2+y^2}$

$\boxed{2.7}$ $u_x + u_y = u^2$ $u(x,0) = 1 \Rightarrow \begin{cases} x_0(s) = s \\ y_0(s) = 0 \\ u_0(s) = 1 \end{cases}$

$\frac{\partial x}{\partial \tau} = 1 \Rightarrow x = \tau + s$

$\frac{\partial y}{\partial \tau} = 1 \Rightarrow y = \tau$

$\frac{\partial u}{\partial \tau} = u^2 \Rightarrow \int \frac{\partial u}{u^2} = \tau + K$

$\Rightarrow -\frac{1}{u} = \tau + K \Rightarrow u = -\frac{1}{\tau + K}$

when $\tau = 0$ $u = 1 \Rightarrow K = -1$

so $u = \frac{1}{1-\tau}$

So the characteristic curves are $\begin{cases} x = \tau + s \\ y = \tau \\ u = \frac{1}{1-\tau} \end{cases}$

and the solution is $u = \frac{1}{1-y}$

Note: the solution is independent of x - Could we have predicted this?

Yes: the initial condition is independent of x and the solution has no explicit terms in x

$\boxed{8.11}$ $(y^2 + u)u_x + yu_y = 0$ in $y > 0$ with $u = 0$ on $x = \frac{y^2}{2}$.

- initial condition curve: take $\begin{cases} x_0(s) = s^2/2 \\ y_0(s) = s \\ u_0(s) = 0. \end{cases}$
- characteristic equations

$$\begin{cases} \frac{\partial x}{\partial \tau} = y^2 + u & (1) \\ \frac{\partial y}{\partial \tau} = y & (2) \\ \partial u/\partial \tau = 0 & (3) \end{cases}$$

From (3) $\Rightarrow u = u_0(s) = 0$

So the solution appears to be 0 "everywhere".

Is this true?

Characteristics are ② $y = se^{\tau}$

① $\frac{\partial x}{\partial \tau} = s^2 e^{2\tau} \Rightarrow x = \frac{s^2}{2} e^{2\tau}$

so $x = \frac{y^2}{2}$

→ so the characteristics are all confounded with the initial condition curve.

⇒ it's a degenerate problem, with an ∞ of solutions

$\boxed{2.12}$ $u_y + u^2 u_x = 0 \qquad x>0 \qquad u(x,0)=\sqrt{x}$

<u>characteristic equations</u> $\Rightarrow \begin{cases} x_0(s) = s \\ y_0(s) = 0 \\ u_0(s) = \sqrt{s} \end{cases}$ for $s>0$

$\frac{\partial y}{\partial \tau} = 1 \quad \Rightarrow \quad y = \tau + y_0(s) = \tau$

$\frac{\partial x}{\partial \tau} = u^2 \quad \Rightarrow \quad \frac{\partial x}{\partial \tau} = s \quad \Rightarrow \quad x = s\tau + s$

$\frac{\partial u}{\partial \tau} = 0 \quad \Rightarrow \quad u = u_0(s) = \sqrt{s}$

So $\quad x = s\,y + s \qquad y = \frac{x-s}{s}$

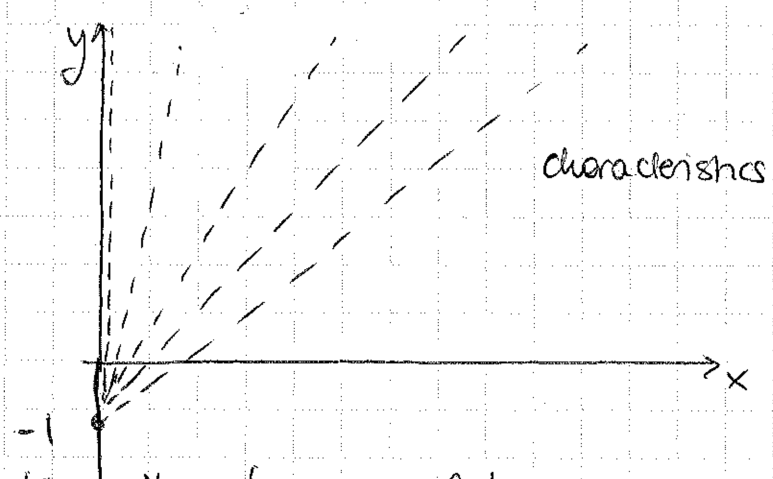
we can solve for s in terms of x and y:

$$s = \frac{x}{y+1}.$$

for $\sqrt{s} > 0$ we require the + solution.

so $\quad u = \sqrt{\frac{x}{y+1}}$

We see that the solution is defined everywhere for $x>0$



but not for $y<-1$

This makes sense based on the characteristics

Note: the transversality condition reads in this case

$$\begin{vmatrix} s & 1 \\ 1 & 0 \end{vmatrix} = -1 \quad \Rightarrow \text{ nowhere } = 0.$$

$\boxed{2.16}$ $\quad xu_x + yu_y = -u$ $\qquad \begin{cases} x_0(s) = \cos s \\ y_0(s) = \sin s \\ u_0(s) = 1 \end{cases} \quad 0 \le s \le \pi$.

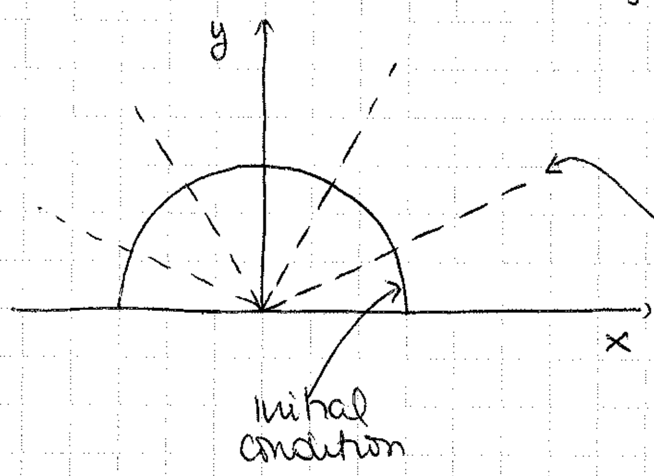
$$\frac{\partial x}{\partial \tau} = x \qquad x = \cos s\, e^{\tau}$$

$$\frac{\partial y}{\partial \tau} = y \qquad y = \sin s\, e^{\tau}$$

$$\frac{\partial u}{\partial \tau} = -u \qquad u = e^{-\tau}$$

so $x^2 + y^2 = e^{2\tau}$ $\qquad$ so $u = \dfrac{1}{\sqrt{x^2+y^2}}$

$\rightarrow$ not defined at $(0,0)$



Note
characteristics are

$$y = \sin(s)\frac{x}{\cos(s)} = \tan(s)\, x$$

$\boxed{2.17}$ $\quad xu_x + u_y = 1$

- Find a characteristic curve through $(1,1,1)$

$$\frac{\partial u}{\partial \tau} = 1 \quad \Rightarrow \quad u = \tau + u_0(s)$$

$$\frac{\partial x}{\partial \tau} = x \quad \Rightarrow \quad x = x_0(s) e^{\tau}$$

$$\frac{\partial y}{\partial \tau} = 1 \quad \Rightarrow \quad y = \tau + y_0(s)$$

if $x_0(s) = 1$, $y_0(s) = 1$ and $u_0(s) = 1$ then

$$\begin{cases} x = e^{\tau} \\ y = \tau \\ u = \tau \end{cases}$$ is the parametric equation for the characteristic curve.

(b) if $u(x,0) = \sin x$ then

$x_0(s) = s$
$y_0(s) = 0$
$u_0(s) = \sin s$

$$\Rightarrow \begin{cases} x = se^{\tau} \\ y = \tau \\ u = \tau + \sin(s) \end{cases}$$

so $x = se^{y} \Rightarrow s = xe^{-y}$

so $u = y + \sin\left(xe^{-y}\right)$

the solution is defined for all x and y

$\boxed{2.18}$ $uu_x + u_y = -\frac{1}{2}u$ $u(x,2x) = x^2$

$$\begin{cases} x_0(s) = s \\ y_0(s) = 2s \\ u_0(s) = s^2 \end{cases}$$

so ③ $\frac{\partial x}{\partial \tau} = u \Rightarrow \frac{\partial x}{\partial \tau} = s^2 e^{-\frac{1}{2}\tau} \Rightarrow x = -2s^2 e^{-\frac{1}{2}\tau} + s + 2s^2$

① $\frac{\partial y}{\partial \tau} = 1 \Rightarrow y = \tau + 2s$

② $\frac{\partial u}{\partial \tau} = -\frac{1}{2}u \Rightarrow u = s^2 e^{-\frac{1}{2}\tau}$

So $\tau = y - 2s \Rightarrow x = -2s^2 e^{-\frac{1}{2}(y-2s)} + s + 2s^2$

Possible but difficult to invert for s

However we can identify the condition for existence/uniqueness of solutions from the transversality condition as

$a(x_0, y_0, u_0) = s^2$ $\quad \frac{dx_0}{ds} = 1$
$b(x_0, y_0, u_0) = 1$ $\quad \frac{dy_0}{ds} = 2$

$$\Rightarrow \begin{vmatrix} a & dx_0/ds \\ b & dy_0/ds \end{vmatrix} = 2s^2 - 1 \Rightarrow \text{problem when } s = \pm\sqrt{\tfrac{1}{2}}.$$