HOMEWORK 1

Partial Differentiation

5.1, 5.5 → see answers in RHB

5.8 $\quad s = \frac{1}{2}(x+y) \qquad t = \frac{1}{2}(x-y)$

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \text{in } (s,t) \text{ coordinate system}$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial \phi}{\partial t}\frac{\partial t}{\partial x} = \frac{1}{2}\frac{\partial \phi}{\partial s} + \frac{1}{2}\frac{\partial \phi}{\partial t} = \frac{1}{2}\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right)\phi$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial \phi}{\partial t}\frac{\partial t}{\partial y} = \frac{1}{2}\frac{\partial \phi}{\partial s} - \frac{1}{2}\frac{\partial \phi}{\partial t} = \frac{1}{2}\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)\phi$$

$$\begin{aligned}\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2} &= \frac{1}{2}\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right)\left[\frac{1}{2}\left(\frac{\partial \phi}{\partial s} + \frac{\partial \phi}{\partial t}\right)\right] \\ &- \frac{1}{2}\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)\left[\frac{1}{2}\left(\frac{\partial \phi}{\partial s} - \frac{\partial \phi}{\partial t}\right)\right] \\ &= \frac{1}{4}\left(\frac{\partial^2 \phi}{\partial s^2} + 2\frac{\partial^2 \phi}{\partial s \partial t} + \frac{\partial^2 \phi}{\partial t^2}\right) \\ &- \frac{1}{4}\left(\frac{\partial^2 \phi}{\partial s^2} - 2\frac{\partial^2 \phi}{\partial s \partial t} + \frac{\partial^2 \phi}{\partial t^2}\right) \\ &= \frac{1}{2}\frac{\partial^2 \phi}{\partial s \partial t} + \frac{1}{2}\frac{\partial^2 \phi}{\partial s \partial t} = \frac{\partial^2 \phi}{\partial s \partial t}.\end{aligned}$$

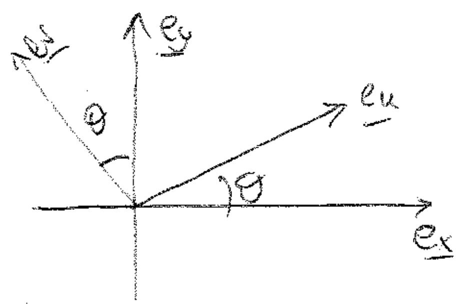
$$\frac{\partial^2 \phi}{\partial s \partial t} = 0 \implies \begin{aligned}\phi(s,t) &= f(s) + g(t) \\ &= \tilde{f}(x+y) + \tilde{g}(x-y)\end{aligned}$$

(f and $\tilde{f}$ slightly different since $s = \frac{1}{2}(x+y)$ ...)

Multivariate calculus

① If you can't do this one please retake 211!

②



$\underline{e_u} = \cos\theta\, \underline{e_x} + \sin\theta\, \underline{e_y}$

$\underline{e_v} = -\sin\theta\, \underline{e_x} + \cos\theta\, \underline{e_y}$

so if a point P has coordinates $\begin{pmatrix} x \\ y \end{pmatrix}$ in $(\underline{e_x}, \underline{e_y})$ system and $\begin{pmatrix} u \\ v \end{pmatrix}$ in $(\underline{e_u}, \underline{e_v})$ system

$$\underline{OP} = x\underline{e_x} + y\underline{e_y}$$

$$= u\underline{e_u} + v\,\underline{e_v}$$

$$= u\left(\cos\theta\, \underline{e_x} + \sin\theta\, \underline{e_y}\right) + v\left(-\sin\theta\, \underline{e_x} + \cos\theta\, \underline{e_y}\right)$$

$$\Rightarrow \begin{cases} x = u\cos\theta - v\sin\theta \\ y = u\sin\theta + v\cos\theta \end{cases}$$

$$\frac{\partial}{\partial u} = \frac{\partial}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\frac{\partial y}{\partial u} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial v} = \frac{\partial}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\frac{\partial y}{\partial v} = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}$$

$$\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} = \left(\cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}\right)\left(\cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}\right)$$
$$+ \left(-\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}\right)\left(-\sin\theta\frac{\partial f}{\partial x} + \cos\theta\frac{\partial f}{\partial y}\right)$$

$$= \cos^2\theta\frac{\partial^2 f}{\partial x^2} + \sin^2\theta\frac{\partial^2 f}{\partial y^2} + 2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y}$$
$$+ \sin^2\theta\frac{\partial^2 f}{\partial x^2} + \cos^2\theta\frac{\partial^2 f}{\partial y^2} - 2\cos\theta\sin\theta\frac{\partial^2 f}{\partial x\partial y}$$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

Vector calculus

(1)

$$F(x) = \begin{pmatrix} x^2 - 3x + \ln z \\ 2x^4 + e^y \\ \sin(xz) \end{pmatrix}$$

$$\nabla \cdot F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$= 2x - 3 + e^y + x\cos(xz)$$

$$\nabla \times F = \begin{pmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{z} - z\cos(xz) \\ 8x^3 \end{pmatrix}$$

(2)

$$\nabla \left(\frac{zx^2}{x^2+y^2+z^2} \right) = \begin{pmatrix} \frac{2xz}{x^2+y^2+z^2} - \frac{2zx^3}{(x^2+y^2+z^2)^2} \\ -\frac{2zyx^2}{(x^2+y^2+z^2)^2} \\ \frac{x^2}{x^2+y^2+z^2} - \frac{2z^2x^2}{(x^2+y^2+z^2)^2} \end{pmatrix}$$

(3) • RHB 10.13 → see hint in textbook

let $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

$$\nabla \cdot (a \times b) = (a_2 b_3 - a_3 b_2)_x + (a_3 b_1 - a_1 b_3)_y + (a_1 b_2 - a_2 b_1)_z$$

$$= a_{2x} b_3 + a_2 b_{3x} - a_{3x} b_2 - a_3 b_{2x} + \dots + \dots$$

similar terms

$= b_1 (a_{3y} - a_{2z}) + b_2 (a_{1z} - a_{3x}) + b_3 (a_{2x} - a_{1y})$

$- a_1 (b_{3y} - b_{2z}) - a_2 (b_{1z} - b_{3x}) - a_3 (b_{2x} - b_{1y})$

$= b \cdot (\nabla \times a) - a \cdot (\nabla \times b)$ as required.

- RHB 10.14

(a) $\nabla \times (a \nabla \cdot a) + a \times [\nabla \times \nabla \times a] + a \times (\nabla^2 a)$

$\rightarrow$ use formulary.

$\nabla \times (A f) = f \nabla \times A + \nabla f \times A$ (eq. 8)

$\Rightarrow \nabla \times (a \nabla \cdot a) = \nabla \cdot a \, \nabla \times a + \nabla (\nabla \cdot a) \times a$

Also $\nabla \times \nabla \times a + \nabla^2 a = \nabla (\nabla \cdot a)$ (eq. 14).

$\Rightarrow \nabla \times (a \nabla \cdot a) + a \times [\nabla \times \nabla \times a + \nabla^2 a]$

$= \nabla \cdot a \, \nabla \times a + \underbrace{\nabla (\nabla \cdot a) \times a + a \times (\nabla (\nabla \cdot a))}_{= 0}$.

$= (\nabla \cdot a) \nabla \times a$.

(b) Similar to RHB 10.13 $\rightarrow$ do it as exercice.

(c) Use eq. 12

$\nabla (\underline{a} \cdot \underline{a}) = 2 a \times (\nabla \times a) + 2 a \cdot \nabla a$

$\Rightarrow a \times (\nabla \times a) = \frac{1}{2} \nabla (a^2) - a \cdot \nabla a$

RHB 10.15 $\rightarrow$ see answers in RHB.

- if $\underline{r}$ is position vector $\underline{r} = (x, y, z)$ in Cartesian

$\nabla \cdot \underline{r} = 3$ $\qquad \underline{r} = (r, 0, 0)$ in Spherical

$\qquad\qquad\qquad \underline{r} = (r, 0, z)$ in cylindrical

$\nabla \times \underline{r} = 0$

$\rightarrow$ answer is the same in all 3 systems (of course).

e.g.
$$\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$
$$\frac{1}{r}\frac{\partial}{\partial r}(r^2) + \frac{\partial z}{\partial z} = 3$$
$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^3) = 3.$$

- $$\nabla \cdot \left(\frac{\underline{r}}{r^3}\right) = \frac{1}{r^3}\nabla \cdot \underline{r} + \nabla\left(\frac{1}{r^3}\right) \cdot \underline{r}$$
$$= \frac{3}{r^3} - \frac{3}{r^4}\underline{e}_r \cdot \underline{r} = 0$$

$$\nabla \times \left(\frac{\underline{r}}{r^3}\right) = \frac{1}{r^3}\nabla \times \underline{r} + \nabla\left(\frac{1}{r^3}\right) \times \underline{r}$$
$$= 0 + \left(-\frac{3}{r^4}\underline{e}_r\right) \times \underline{r} = 0$$

RHB 14.2

- $$y' = xy^3$$

$$\frac{y'}{y^3} = x$$

$$\frac{dy}{y^3} = x\,dx \rightarrow \frac{y^{-2}}{-2} = \frac{x^2}{2} + k$$

$$-\frac{1}{y^2} - x^2 = c \Rightarrow y^2 = -\frac{1}{c+x^2}$$

$\rightarrow$ solution exists when $c+x^2<0$

- $$y' \tan^{-1}x - \frac{y}{1+x^2} = 0$$

$$\frac{dy}{y} = \frac{dx}{1+x^2}\,\frac{1}{\tan^{-1}x}$$

let $u = \tan^{-1}x$

$$du = \frac{1}{1+x^2}\,dx \quad \text{So}$$

$$\frac{dy}{y} = \frac{du}{u} \Rightarrow \ln y = \ln u + k$$

$$y = cu$$

$$= C\tan^{-1}x$$

- $$x^2y' + xy^2 = 4y^2$$

$$\frac{dy}{y^2} = \frac{4-x}{x^2}\,dx$$

$$-\frac{1}{y} = -\frac{4}{x} - \ln x + k$$

$$y = \frac{1}{\frac{4}{x} + \ln x + c}$$

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- $(1-x^2)\, y' + 2xy = (1-x^2)^{3/2}$

$$y' + \frac{2x}{1-x^2}\, y = (1-x^2)^{1/2}$$

$$\mu = e^{\int \frac{2x}{1-x^2}\,dx}$$

$$= e^{-\ln(1-x^2)} = \frac{1}{1-x^2}$$

so $\frac{\partial}{\partial x}\left(\frac{y}{1-x^2}\right) = \frac{1}{\sqrt{1-x^2}}$

$$\frac{y}{1-x^2} = \int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + K$$

$$y = (1-x^2)\arcsin(x) + K(1-x^2)$$

- $y' - y\cot x = -\operatorname{cosec} x$

$$y' - y\,\frac{\cos x}{\sin x} = -\frac{1}{\sin x}$$

$$p = e^{-\int \frac{\cos x}{\sin x}\,dx} = e^{-\ln \sin x} = \frac{1}{\sin x}$$

$$\frac{\partial}{\partial x}\left(\frac{y}{\sin x}\right) = -\frac{1}{\sin^2 x}$$

$$\frac{y}{\sin x} = \int -\frac{dx}{\sin^2 x} = \cot x + K$$

$$y = \sin x\left(\frac{\cos x}{\sin x} + K\right) = \cos x + K\sin x$$

- $(x+y^3)y' = y$

$\rightarrow \quad \frac{dy}{dx}(x+y^3) = y$ (note that's not linear, hence the hint)

$$y\,\frac{dx}{dy} = x + y^3$$

$$\frac{dx}{dy} - \frac{x}{y} = y^2 \qquad p = e^{-\int \frac{dy}{y}} = e^{-\ln y} = \frac{1}{y}$$

$$\Rightarrow \quad \frac{d}{dy}\left(\frac{x}{y}\right) = \frac{y^2}{y} = y$$

$$\frac{x}{y} = \int y^2 \, dy = \frac{y^3}{3} + K$$

$$x = \frac{y^4}{3} + Ky.$$

$\boxed{\text{PHB } 14.6}$

$$\frac{dy}{dx} = -\frac{2x^2+y^2+x}{xy}$$

Hint/trick: use $Y = y^2 \Rightarrow y\frac{dy}{dx} = -\left(2x + \frac{y^2}{x} + 1\right)$

$$\Rightarrow \quad \frac{1}{2}\frac{dY}{dx} = -2x - \frac{Y}{x} - 1$$

$$\Rightarrow \quad \frac{dY}{dx} = -(4x+2) - \frac{2Y}{x}$$

$$\Rightarrow \quad \mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$$

$$\Rightarrow \quad \frac{d}{dx}\left(x^2 Y\right) = -\left(4x^3 + 2x^2\right)$$

$$x^2 Y = -x^4 - \frac{2}{3}x^3 + K$$

$$Y = -x^2 - \frac{2}{3}x + \frac{K}{x^2}$$ $\Rightarrow$ solution exists when $Y > 0$.

$$\Rightarrow \quad y = \pm\sqrt{-x^2 - \frac{2}{3}x + \frac{K}{x^2}} \quad \text{when } Y > 0.$$

RHB 14.11 $\rightarrow$ see RHB for solution

RHB 14.16

$$\frac{dy}{dx} = \tan x \cos y \,(\cos y + \sin y)$$

Using the hint: $\frac{d}{dy}\left(\ln(1+\tan y)\right)$

$$= \frac{(\tan y)'}{1+\tan y} = \frac{1}{\cos^2 y + \cos^2 y \tan y}$$

$$= \frac{1}{\cos^2 y + \cos y \sin y} = \frac{1}{\cos y(\cos y + \sin y)}$$

$$\rightarrow \quad \frac{dy}{\cos y(\cos y + \sin y)} = \tan x \; dx$$

$$\rightarrow \quad \ln(1+\tan y) = -\ln\cos x + K$$

$$1+\tan y = \frac{K'}{\cos x} \Rightarrow \tan y = \frac{K'}{\cos x} - 1$$

$$\rightarrow \quad y = \tan^{-1}\left[\frac{K'}{\cos x} - 1\right]$$

RHB 14.24

- $y' - \left(\frac{y}{x}\right) = 1 \qquad y(1) = -1$

$$\mu(x) = e^{\int -\frac{1}{x}\,dx} = \frac{1}{x}$$

$$\rightarrow \quad \left(\frac{y}{x}\right)' = \frac{1}{x} \Rightarrow \frac{y}{x} = \ln x + K$$

$\Rightarrow \quad y = x\ln x + kx$

If $y(1) = -1$ then

$-1 = k \cdot 1 \Rightarrow k = -1$

and $\boxed{y = x\ln x - x}$

- $y' - y\tan x = 1 \qquad y(\pi/4) = 3$

$\mu(x) = e^{\int -\tan x\,dx} = e^{\int -\frac{\sin x}{\cos x}\,dx} = e^{\ln\cos x}$

$= \cos x$

$(y \cdot \cos x)' = \cos x \Rightarrow y\cos x = \sin x + k$

$\Rightarrow y = \tan x + \dfrac{k}{\cos x}$

$3 = \tan\left(\frac{\pi}{4}\right) + \dfrac{k}{\cos(\pi/4)} = 1 + \dfrac{k}{\frac{\sqrt{2}}{2}}$

$\Rightarrow k = 2 \cdot \dfrac{\sqrt{2}}{2} = \sqrt{2}$

So $\boxed{y = \tan x + \dfrac{\sqrt{2}}{\cos x}}$

- $y' - \dfrac{y^2}{x^2} = \dfrac{1}{4} \qquad y(1) = 1 \quad$ and $\quad y(1) = \frac{1}{2}$

$\rightarrow$ Homogeneous equation let $u = \dfrac{y}{x}$

$\rightarrow x u' + u - u^2 = \dfrac{1}{4} \quad \Rightarrow u' = \dfrac{\frac{1}{4} - v + v^2}{x}$

$$\frac{du}{\frac{1}{4} - u + u^2} = x^{-1}\,dx$$

$$\Rightarrow \quad \ln x = \int \frac{du}{\frac{1}{4} - u + u^2} = \int \frac{du}{\left(u - \frac{1}{2}\right)^2}$$

$\rightarrow$ Let $v = u - \frac{1}{2}$ then

$$\ln x = \int \frac{dv}{v^2} = K - \frac{1}{v} = K - \frac{1}{u - \frac{1}{2}} = K - \frac{1}{\frac{y}{x} - \frac{1}{2}}.$$

So $$\frac{1}{\frac{y}{x} - \frac{1}{2}} = K - \ln x \Rightarrow \frac{y}{x} - \frac{1}{2} = \frac{1}{K - \ln x}$$

$$y = x\left[\frac{1}{2} + \frac{1}{K - \ln x}\right].$$

(c) If we want $y(1) = 1$

$$1 = 1\left[\frac{1}{2} + \frac{1}{K}\right] \rightarrow K = 2.$$

So $$y = x\left[\frac{1}{2} + \frac{1}{2 - \ln x}\right]$$

(a) If we want $y(1) = 1/2$ $\rightarrow$ no solution.