Solve by power series $(1-x^2)y'' - xy' + n^2y = 0$. The polynomial solutions of this equation with coefficients determined to make $y(1) = 1$ are called Chebyshev polynomials $T_n(x)$. Find T_0, T_1, and T_2.

(a) The following differential equation is often called a Sturm-Liouville equation:

$$\frac{d}{dx}[A(x)y'] + [\lambda B(x) + C(x)]y = 0$$

(λ is a constant parameter). This equation includes many of the differential equations of mathematical physics as special cases. Show that the following equations can be written in the Sturm-Liouville form: the Legendre equation (7.2); Bessel's equation (19.2) for a *fixed* p, that is, with the parameter λ corresponding to a^2; the simple harmonic motion equation $y'' = -n^2y$; the Hermite equation (22.14); the Laguerre equations (22.21) and (22.26).

(b) By following the methods of the orthogonality proofs in Sections 7 and 19, show that if y_1 and y_2 are two solutions of the Sturm-Liouville equation (corresponding to the two values λ_1 and λ_2 of the parameter λ), then y_1 and y_2 are orthogonal on (a, b) with respect to the weight function $B(x)$ if $A(x)(y_1'y_2 - y_2'y_1)|_a^b = 0$.

In Problem 22.26, replace x by x/n in the y differential equation and set $\lambda = n$ to show that the differential equation satisfied by the functions $f_n(x)$ in Problem 22.27 is

$$y'' + \left(\frac{1}{x} - \frac{1}{4n^2} - \frac{l(l+1)}{x^2}\right)y = 0.$$

Hence show by Problem 24 that the functions $f_n(x)$ are orthogonal on $(0, \infty)$.

Verify *Bauer's formula* $e^{ixw} = \sum_0^\infty (2l+1)i^l j_l(x)P_l(w)$ as follows. Write the integral for the coefficients c_l in the Legendre series for $e^{ixw} = \sum c_l P_l(w)$. You want to show that $c_l(x) = (2l+1)i^l j_l(x)$. First show that $y = c_l(x)$ satisfies the differential equation (Problem 17.6) for spherical Bessel functions. *Hints:* Differentiate with respect to x under the integral sign to find y' and y''; substitute into the left side of the differential equation. Now integrate by parts with respect to w to show that the integrand is zero because $P_l(w)$ satisfies Legendre's equation. Thus $c_l(x)$ must be a linear combination of $j_l(x)$ and $n_l(x)$. Now consider the $c_l(x)$ integral for small x; expand e^{ixw} in series and evaluate the lowest term (which is x^l since $\int_{-1}^1 w^n P_l(w)\,dw = 0$ for $n < l$). Compare with the approximate formulas for $j_l(x)$ and $n_l(x)$ in Section 20.

Verify that the Legendre equation (2.1) can be written in either of the forms

$$RLy_l = -l^2y \qquad \text{or} \qquad LRy_{l-1} = -l^2y_{l-1}$$

where y_l means $P_l(x)$ and R and L are the operators

$$R = (1-x^2)D - lx, \qquad L = (1-x^2)D + lx.$$

Following the method used for Hermite functions [equations (22.1) to (22.10)], show that R and L are raising and lowering operators. In fact, show [by considering the operators when $x = 1$ and requiring $y_l(1) = 1$] that $Ly_l = ly_{l-1}$ and $Ry_{l-1} = -ly_l$. Solve $Ly_0 = 0$ assuming $y_0(1) = 1$ to find $y_0 = P_0(x) = 1$. Use the raising operator equation $Ry_{l-1} = -ly_l$ to find $P_1(x)$ and $P_2(x)$.

13

PARTIAL DIFFERENTIAL EQUATIONS

1. INTRODUCTION

Many of the problems of mathematical physics involve the solution of partial differential equations. The same partial differential equation may apply to a variety of physical problems; thus the mathematical methods which you will learn in this chapter apply to many more problems than those we shall discuss in the illustrative examples. Let us outline the partial differential equations we shall consider, and the kinds of physical problems which lead to each of them.

(1.1) **Laplace's equation** $\quad \nabla^2 u = 0$

The function u may be the gravitational potential in a region containing no matter, the electrostatic potential in a charge-free region, the steady-state temperature (that is, temperature not changing with time) in a region containing no source of heat, or the velocity potential for an incompressible fluid with no vortices and no sources or sinks.

(1.2) **Poisson's equation** $\quad \nabla^2 u = f(x, y, z)$

The function u may represent the same physical quantities listed for Laplace's equation, but in a region containing matter, electric charge, or sources of heat or fluid, respectively, for the various cases. The function $f(x, y, z)$ is called the source density; for example, in electricity it is proportional to the density of electric charge.

) The diffusion or heat flow equation $\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$

Iere u may be the non-steady-state temperature (that is, temperature varying with e) in a region with no heat sources; or it may be the concentration of a diffusing stance (for example, a chemical, or particles such as neutrons). The quantity α^2 is a stant known as the diffusivity.

) Wave equation $\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$

Here u may represent the displacement from equilibrium of a vibrating string or mbrane or (in acoustics) of the vibrating medium (gas, liquid, or solid); in electricity nay be the current or potential along a transmission line; or u may be a component E or H in an electromagnetic wave (light, radio waves, etc.). The quantity v is the ed of propagation of the waves; for example, for light waves in a vacuum it is c, the ocity of light, and for sound waves it is the speed at which sound travels in the dium under consideration.

5) Helmholtz equation $\nabla^2 F + k^2 F = 0$

As you will see later, the function F here represents the space part (that is, the ne-independent part) of the solution of either the diffusion or the wave equation.

We shall be principally concerned with the solution of these equations rather than eir derivation. If you like, you could say that it is true experimentally that the ysical quantities mentioned above satisfy the given equations. However, it is also true at the equations can be derived from somewhat simpler experimental assumptions. et us indicate briefly an example of how this can be done. In Chapter 6, Sections 10 id 11, we considered the flow of fluid. We showed (Chapter 6, Problem 10.15) that · v = 0 for an incompressible fluid in a region containing no sources or sinks. If it is so true that there are no vortices (that is, the flow is irrotational), then curl v = 0, id v can be written as the gradient of a scalar function: $\mathbf{v} = \nabla u$. Combining these vo equations, we have $\nabla \cdot \nabla u = \nabla^2 u = 0$. The function u is called the velocity potenal and we see that (under the given conditions) it satisfies Laplace's equation as we aimed. A few more examples of such derivations are outlined in the problems.

In the following sections, we shall consider a number of physical problems to illusate the very useful method of solving partial differential equations known as *separation f variables* (no relation to the same term used in ordinary differential equations, Chapter 8). In Sections 2 to 4, we consider problems in rectangular coordinates leading) Fourier series solutions—problems similar to those solved by Fourier. In later ections, we consider use of other coordinate systems (cylindrical, spherical) leading to olutions using Legendre or Bessel series.

PROBLEMS, SECTION 1

. ... electricity the equations $\nabla \cdot \mathbf{D} = \rho$ (D = electric displacement and ρ = charge

the electrostatic potential satisfies Laplace's equation (1.1) in a charge-free region and satisfies Poisson's equation (1.2) in a region of charge density ρ.

2. Show that the expression $u = \sin(x - vt)$ describing a sinusoidal wave, satisfies the wave equation. Show that in general

$$u = f(x - vt) \quad \text{and} \quad u = f(x + vt)$$

satisfy the wave equation (1.4), where f is any function with a second derivative.

3. Assume from electricity the following equations which are valid in free space. (They are called Maxwell's equations.)

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{H} = 0,$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}, \qquad \nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}.$$

From them show that any component of E or H satisfies the wave equation (1.4) with $v = (\epsilon\mu)^{-1/2}$. *Hint:* Use vector identity (e) in the table at the end of Chapter 6.

4. Obtain the heat flow equation (1.3) as follows: The quantity of heat Q flowing across a surface is proportional to the normal component of the (negative) temperature gradient $(-\nabla T) \cdot \mathbf{n}$. Compare Chapter 6, equation (10.4), and apply the discussion of flow of water given there to the flow of heat. Thus show that the rate of gain of heat per unit volume per unit time is proportional to $\nabla \cdot \nabla T$. But $\partial T/\partial t$ is proportional to this gain in heat; thus show that T satisfies (1.3).

2. LAPLACE'S EQUATION; STEADY-STATE TEMPERATURE IN A RECTANGULAR PLATE

We want to solve the following problem: A long rectangular metal plate has its two long sides and the far end at 0° and the base at 100° (Figure 2.1). The width of the plate is 10 cm. Find the steady-state temperature distribution inside the plate. (This problem is mathematically identical to the problem of finding the electrostatic potential in the region $0 < x < 10$, $y > 0$, if the given temperatures are replaced by potentials—see, for example, Jackson, p. 72.)

To simplify the problem, we shall assume at first that the plate is so long compared to its width that we may make the mathematical approximation that it extends to infinity in the y direction. It is then called a semi-infinite plate. This is a good assumption if we are interested in temperatures not too near the far end.

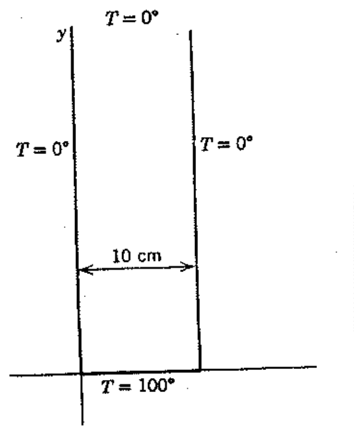


FIGURE 2.1

The temperature T satisfies Laplace's equation inside the plate where there are no sources of heat, that is,

$$\nabla^2 T = 0 \quad \text{or} \quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0. \tag{2.1}$$

We have written ∇^2 in rectangular coordinates be- ... boundary of the plate is rectangular and ve omitted the z term because the plate is in two dimensions. To solve this on, we are going to *try* a solution of the form

$$T(x, y) = X(x)Y(y),$$

, as indicated, X is a function only of x, and Y is a function only of y. Immedi- you may raise the question: But how do we know that the solution is of this The answer is that it is not! However, as you will see, once we have solutions of rm (2.2) we can combine them to get the solution we want. [Note that a sum of ons of (2.1) is a solution of (2.1).] Substituting (2.2) into (2.1), we have

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0.$$

nary instead of partial derivatives are now correct since X depends only on x, etc.) e (2.3) by XY to get

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0.$$

e next step is really the key to the process of *separation of variables.* We are going that each of the terms in (2.4) is a constant because the first term is a function of ne and the second term is a function of y alone. Why is this correct? Recall that we say $u = \sin t$ is a *solution* of $\ddot{u} = -u$, we mean that if we substitute $u = \sin t$ he differential equation, we get an identity ($\ddot{u} = -u$ becomes $-\sin t = -\sin t$), is true for all values of t. Although we speak of an *equation*, when we substitute lution into a differential equation, we have an *identity* in the independent variable. made use of this fact in series solutions of differential equations in Chapter 12, ons 1 and 2.) In (2.1) to (2.4) we have two independent variables, x and y. Saying 2.2) is a solution of (2.1) means that (2.4) is an identity in the two independent bles x and y [recall that (2.4) was obtained by substituting (2.2) into (2.1)]. In words, if (2.2) is a solution of (2.1), then (2.4) must be true for any and all values e two independent variables x and y. Since X is a function only of x and Y of y, rst term of (2.4) is a function only of x and the second term is a function only of ppose we substitute a particular x into the first term; that term is then some erical constant. To have (2.4) satisfied, the second term must be minus the same ant. While x remains fixed, let y vary (remember that x and y are independent). ave said that (2.4) is an identity; it is then true for our fixed x and *any* y. Thus econd term remains constant as y varies. Similarly, if we fix y and let x vary, we hat the first term of (2.4) is a constant. To say this more concisely, the equation $= g(y)$, with x and y independent variables, is an identity only if both functions he same constant; this is the basis of the process of separation of variables. From we then write

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \text{const.} = -k^2, \qquad k \geq 0, \qquad \text{or}$$

The constant k^2 is called the *separation constant.* The solutions of (2.5) are

(2.6) $$X = \begin{cases}\sin kx, \\ \cos kx,\end{cases} \qquad Y = \begin{cases}e^{ky}, \\ e^{-ky},\end{cases}$$

and the solutions of (2.1) of the form (2.2) are

(2.7) $$T = XY = \begin{cases} e^{ky} \sin kx, \\ e^{-ky} \sin kx, \\ e^{ky} \cos kx, \\ e^{-ky} \cos kx. \end{cases}$$

None of these four basic solutions satisfies the given boundary temperatures. What we must do now is to take a combination of the solutions (2.7), with the constant k properly selected, which *will* satisfy the given boundary conditions. [Any linear combination of solutions of (2.1) is a solution of (2.1) because the differential equation (2.1) is *linear*; see Chapter 3, Section 7, and Chapter 8, Sections 1 and 6.] We first discard the solutions containing e^{ky} since we are given $T \to 0$ as $y \to \infty$. (We are assuming $k > 0$; see Problem 5.) Next we discard solutions containing $\cos kx$ since $T = 0$ when $x = 0$. This leaves us just $e^{-ky} \sin kx$, but the value of k is still to be determined. When $x = 10$, we are to have $T = 0$; this will be true if $\sin(10k) = 0$, that is, if $k = n\pi/10$ for $n = 1, 2, \cdots$. Thus for any integral n, the solution

(2.8) $$T = e^{-n\pi y/10} \sin \frac{n\pi x}{10}$$

satisfies the given boundary conditions on the three $T = 0$ sides.

Finally, we must have $T = 100$ when $y = 0$; this condition is not satisfied by (2.8) for any n. But a linear combination of solutions like (2.8) is a solution of (2.1); let us try to find such a combination which does satisfy $T = 100$ when $y = 0$. In order to allow all possible n's we write an infinite series for T, namely

(2.9) $$T = \sum_{n=1}^{\infty} b_n e^{-n\pi y/10} \sin \frac{n\pi x}{10}.$$

For $y = 0$, we must have $T = 100$; from (2.9) with $y = 0$ we get

(2.10) $$T_{y=0} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{10} = 100.$$

But this is just the Fourier sine series (Chapter 7, Section 9) for $f(x) = 100$ with $l = 10$. We can find the coefficients b_n as we did in Chapter 7; we get

(2.11) $$b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\, dx = \frac{2}{10}\int_0^{10} 100 \sin \frac{n\pi x}{10}\, dx$$

$$= 20 \cdot \frac{10}{n\pi}\left(-\cos \frac{n\pi x}{10}\right)\Bigg|_0^{10} = -\frac{200}{n\pi}[(-1)^n - 1] = \begin{cases}\dfrac{400}{n\pi}, & \text{odd } n, \\ 0, & \text{even } n.\end{cases}$$

hen (2.9) becomes

$$T = \frac{400}{\pi}\left(e^{-\pi y/10}\sin\frac{\pi x}{10} + \frac{1}{3}e^{-3\pi y/10}\sin\frac{3\pi x}{10} + \cdots\right). \tag{2.12}$$

quation (2.12) can be used for computation if $\pi y/10$ is not too small since then the ries converges rapidly. (See also Problem 6.) For example, at $x = 5$ (central line of e plate) and $y = 5$, we find

$$\begin{aligned} T &= \frac{400}{\pi}\left(e^{-\pi/2}\sin\frac{\pi}{2} + \frac{1}{3}e^{-3\pi/2}\sin\frac{3\pi}{2} + \cdots\right) \\ &= \frac{400}{\pi}(0.208 - 0.003 + \cdots) = 26.1°. \end{aligned} \tag{2.13}$$

If the temperature on the bottom edge is any function $f(x)$ instead of 100° (with the her three sides at 0° as before), we can do the problem by the same method. We have ly to expand the given $f(x)$ in a Fourier sine series and substitute the coefficients to (2.9).

Next, let us consider a finite plate of height 30 cm with the top edge at $T = 0°$, and her dimensions and temperatures as in Figure 2.1. We no longer have any reason to scard the e^{ky} solution since y does not become infinite. We now replace e^{-ky} by a near combination $ae^{-ky} + be^{ky}$ which is zero when $y = 30$. The most convenient way do this is to use the combination

$$\tfrac{1}{2}e^{k(30-y)} - \tfrac{1}{2}e^{-k(30-y)} \tag{2.14}$$

hat is, let $a = \frac{1}{2}e^{30k}$ and $b = -\frac{1}{2}e^{-30k}$). Then, when $y = 30$, (2.14) gives $e^0 - e^0 = 0$ we wanted. Now (2.14) is just $\sinh k(30 - y)$ (see Chapter 2, Section 12), so for the ite plate, we can write the solution as [compare (2.9)]

$$T = \sum_{n=1}^{\infty} B_n \sinh\frac{n\pi}{10}(30 - y)\sin\frac{n\pi x}{10}. \tag{2.15}$$

ch term of this series is zero on the three $T = 0$ sides of the plate. When $y = 0$, we ant $T = 100$:

$$T_{y=0} = 100 = \sum_{n=1}^{\infty} B_n \sinh(3n\pi)\sin\frac{n\pi x}{10} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{10} \tag{2.16}$$

here $b_n = B_n \sinh 3n\pi$ or $B_n = b_n/\sinh 3n\pi$. We find b_n, solve for B_n and substitute to (2.15) to get the temperature distribution in the finite plate:

$$T = \sum_{\text{odd } n} \frac{400}{n\pi \sinh 3n\pi}\sinh\frac{n\pi}{10}(30 - y)\sin\frac{n\pi x}{10}. \tag{2.17}$$

In (2.12) and (2.17) we have found functions $T(x, y)$ satisfying both (2.1) and all the ven boundary conditions. For a bounded region with given boundary temperatures, it an experimental fact (and it can also be shown mathematically—see Problem 16 and hapter 14, Problem 11.38) that there is only one $T(x, y)$ satisfying Laplace's equation d the given boundary condition. Thus (2.17) is the desired solution for the rectangu-

lar plate. It can also be shown that there is only one solution for the semi-infinite plate provided $T \to 0$ at ∞; thus (2.12) is the solution for that case.

It may have occurred to you to wonder why we took the constant in (2.5) to be $-k^2$ and what would happen if we took $+k^2$ instead. As far as getting solutions of the differential equation is concerned it would be perfectly correct to use $+k^2$; we would get instead of (2.7):

$$T = XY = \begin{cases} e^{kx}\sin ky, \\ e^{-kx}\sin ky, \\ e^{kx}\cos ky, \\ e^{-kx}\cos ky. \end{cases} \tag{2.18}$$

[We are assuming that k is real; an imaginary k in (2.18) would simply give combinations of the solutions (2.7) over again. Also see Problem 5.] The solutions (2.18) would not be of any use for the semi-infinite plate problem since none of them tends to zero as $y \to \infty$, and a linear combination of e^{kx} and e^{-kx} cannot be zero both at $x = 0$ and at $x = 10$. However, if we had considered a semi-infinite plate with its long sides parallel to the x axis instead of the y axis, and $T = 100°$ along the short end on the y axis, the solutions (2.18) would have been the ones needed. Or, for the finite plate, if the 100° side were along the y axis, then we would want (2.18).

Finally, let us see how to find the temperature distribution in a plate if two adjacent sides are held at 100° and the other two at 0° (or, in general, if any values are given for the four sides). We can find the solution to this problem by a combination of the results we have already obtained. Let us call the sides of the rectangular plate A, B, C, D (Figure 2.2). If sides A, B, and C are held at 0°, and D at 100°, we can find the temperature distribution by the same method we used in finding (2.17) if we take the x axis along D. Next suppose that for the same plate (Figure 2.2) sides A, B, and D are held at 0° and C at 100°. This is the same kind of problem over again, but this time we want to use the basic solutions (2.18). [Or to shortcut the work, we could write the solution like (2.17) with the x axis taken along C and then interchange x and y in the result to agree with Figure 2.2.] Having obtained the two solutions (one for C at 100° and one for D at 100°), let us add these two answers. The result is a solution of the differential equation (2.1) (linearity: the sum of any two solutions is a solution). The temperatures on the boundary (as well as inside) are the sums of the temperatures in the two solutions we added, that is, 0° on A, 0° on B, 0° + 100° on C, and 100° + 0° on D. These are the given boundary conditions we wanted to satisfy. Thus the sum of the solutions of two simple problems gives the answer to the more complicated one (see Problems 11 to 13).

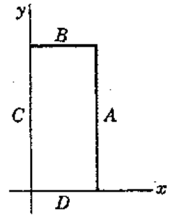


FIGURE 2.2

Before solving more problems, let us stop for a moment to summarize this process of separation of variables which is basically the same for all the partial differential equations we shall discuss. We first assume a solution which is a product of functions of the independent variables [like (2.2)], and separate the partial differential equation into several ordinary differential equations [like (2.5)]. We solve these ordinary differential equations: the solutions may be exponential functions, trigonometric functions, powers

ositive or negative), Bessel functions, Legendre polynomials, etc. Any linear combina-
on of these basic solutions, with any values of the separation constants, is a solution of
e differential equation. The problem is to determine both the values of the separation
nstants and the correct linear combination to fit the given boundary or initial condi-
ons.

The problem of finding the solution of a given differential equation subject to given
undary conditions is called a *boundary value problem*. Such problems often lead to
envalue problems. Recall (Chapter 10, Section 4, and Chapter 12, end of Section 2)
at in an eigenvalue (or characteristic value) problem, there is a parameter whose
lues are to be selected so that the solutions of the problem meet some given require-
ents. The separation constants we have been using are just such parameters; their
lues are determined by demanding that the solutions satisfy some of the boundary
nditions. [For example, we found $k = n\pi/10$ just before (2.8) by requiring that $T = 0$
en $x = 10$.] The resulting values of the separation constants are called *eigenvalues*
d the basic solutions of the differential equation [for example, (2.8)] corresponding to
eigenvalues are called *eigenfunctions*. It may also happen that in addition to the
paration constants there is a parameter in the original partial differential equation (for
ample, ϵ in the Schrödinger equation in Problem 7.17). Again, the possible values of
s parameter for which the equation has solutions meeting specified requirements, are
led eigenvalues, and the corresponding solutions are called eigenfunctions.

OBLEMS, SECTION 2

Find the steady-state temperature distribution for the semi-infinite plate problem if the temperature of the bottom edge is $T = f(x) = x$ (in degrees; that is, the temperature at x cm is x degrees), the temperature of the other sides is 0°, and the width of the plate is 10 cm.

Answer: $T = \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n\pi y/10}\sin(n\pi x/10)$.

Solve the semi-infinite plate problem if the bottom edge of width 20 is held at

$$T = \begin{cases} 0^\circ, & 0 < x < 10, \\ 100^\circ, & 10 < x < 20, \end{cases}$$

and the other sides are at 0°.

Solve the semi-infinite plate problem if the bottom edge of width π is held at $T = \cos x$, and the other sides are at 0°.

Answer: $T = \frac{4}{\pi}\sum_{\text{even } n}\frac{n}{n^2-1}e^{-ny}\sin nx$.

Solve the semi-infinite plate problem if the bottom edge of width 30 is held at

$$T = \begin{cases} x, & 0 < x < 15, \\ 30 - x, & 15 < x < 30, \end{cases}$$

and the other sides are at 0°.

5. Show that the solutions of (2.5) can also be written as

$$X = \begin{cases} e^{ikx}, \\ e^{-ikx}, \end{cases} \qquad Y = \begin{cases} \sinh ky, \\ \cosh ky. \end{cases}$$

Also show that these solutions are equivalent to (2.7) if k is real and equivalent to (2.18) if k is pure imaginary. (See Chapter 2, Section 12.) Also show that $X = \sin k(x - a)$, $Y = \sinh k(y - b)$ are solutions of (2.5).

6. Show that the series in (2.12) can be summed to get

$$T = \frac{200}{\pi}\arctan\left(\frac{\sin(\pi x/10)}{\sinh(\pi y/10)}\right)$$

(with the arc tangent in radians). Use this formula to check the value $T = 26.1°$ at $x = y = 5$. *Hints for summing the series:* Use $\sin(n\pi x/10) = \text{Im}\, e^{in\pi x/10}$ to write the series as $\text{Im}\sum_{\text{odd } n} z^n/n$. (What is z?) Compare this with the series for $\ln[(1+z)/(1-z)]$ (see Chapter 1, Problem 13.22). Then use (13.5) of Chapter 2.

7. Solve Problem 3 if the plate is cut off at height 1 and the temperature at $y = 1$ is held at 0°.

Answer: $T = \frac{4}{\pi}\sum_{\text{even } n}\frac{n}{(n^2-1)\sinh n}\sinh n(1-y)\sin nx$.

8. Find the steady-state temperature distribution in a rectangular plate 30 cm by 40 cm given that the temperature is 0° along the two long sides and along one short end; the other short end along the x axis has temperature

$$T = \begin{cases} 100^\circ, & 0 < x < 10, \\ 0^\circ, & 10 < x < 30. \end{cases}$$

9. Solve problem 2 if the plate is cut off at height 10 and the temperature of the top edge is 0°.

10. Find the steady-state temperature distribution in a metal plate 10 cm square if one side is held at 100° and the other three sides at 0°. Find the temperature at the center of the plate.

Answer: $T = \sum_{\text{odd } n}\frac{400}{n\pi\sinh n\pi}\sinh\frac{n\pi}{10}(10-y)\sin\frac{n\pi x}{10}$,

$T(5, 5) \cong 25°$.

11. Find the steady-state temperature distribution in the plate of Problem 10 if two adjacent sides are at 100° and the other two at 0°. *Hint:* Use your solution of Problem 10. You should not have to do any calculation—just write the answer!

12. Find the temperature distribution in a rectangular plate 10 cm by 30 cm if two adjacent sides are held at 100° and the other two sides at 0°.

13. Find the steady-state temperature distribution in a rectangular plate covering the area $0 < x < 10$, $0 < y < 20$, if the two adjacent sides along the axes are held at temperatures $T = x$ and $T = y$ and the other two sides at 0°.

14. In the rectangular plate problem, we have so far had the temperature specified all around the boundary. We could, instead, have some edges insulated. The heat flow across an edge is proportional to $\partial T/\partial n$, where n is a variable in the direction normal to the edge (see normal derivatives, Chapter 6, Section 6). For example, the heat flow across an edge lying

along the x axis is proportional to $\partial T/\partial y$. Since the heat flow across an insulated edge is zero, we must have not T, but a partial derivative of T, equal to zero on an insulated boundary. Use this fact to find the steady-state temperature distribution in a semi-infinite plate of width 10 cm if the two long sides are insulated, the far end (at ∞ as in Section 2) s at 0°, and the bottom edge is at $T = f(x) = x - 5$.

Note that you used $T \to 0$ as $y \to \infty$ only to discard the solutions e^{+ky}; it would be just as satisfactory to say that T does not become infinite as $y \to \infty$. Actually, the temperature assumed finite) as $y \to \infty$ in this problem is determined by the given temperature at $y = 0$. Let $T = f(x) = x$ at $y = 0$, repeat your calculations above to find the temperature distribu- ion, and find the value of T for large y. Don't forget the $k = 0$ term in the series!

Consider a finite plate, 10 cm by 30 cm, with two insulated sides, one end at 0° and the other at a given temperature $T = f(x)$. Try $f(x) = 100°$; $f(x) = x$. You should convince ourself that this problem cannot be done using just the solutions (2.7). To see what is wrong, go back to the differential equations (2.5) and solve them if $k = 0$. You should find olutions x, y, xy, and constant [the constant is already contained in (2.7) for $k = 0$, but the other three solutions are not]. Now go back over each of the problems we have done so far nd see why we could ignore these $k = 0$ solutions; then including the $k = 0$ solutions, nish the problem of the finite plate with insulated sides.

For the case $f(x) = x$, the answer is:

$$T = \tfrac{1}{6}(30 - y) - \frac{40}{\pi^2} \sum_{\text{odd } n} \frac{1}{n^2 \sinh 3n\pi} \sinh \frac{n\pi}{10}(30 - y) \cos \frac{n\pi x}{10}.$$

how that there is only one function u which takes given values on the (closed) boundary of region and satisfies Laplace's equation $\nabla^2 u = 0$ in the interior of the region. *Hints:* uppose u_1 and u_2 are both solutions with the same boundary conditions so that $U = {}_1 - u_2 = 0$ on the boundary. In Green's first identity (Chapter 6, Problem 10.16), t $\phi = \psi = U$ to show that $\nabla U \equiv 0$. Thus show $U \equiv 0$ everywhere inside the region.

IE DIFFUSION OR HEAT FLOW EQUATION; FLOW IN A BAR OR SLAB

eat flow equation is

$$\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t},$$

u is the temperature and α^2 is a constant characteristic of the material through the heat is flowing. It is worth while to do first a partial separation of (3.1) into equation and a time equation; the space equation in more than one dimension ust be further separated into ordinary differential equations in x and y, or x, y, or r, θ, ϕ, etc. We assume a solution of (3.1) of the form

$$u = F(x, y, z)T(t).$$

he change in meaning of T; we have previously used it for temperature; here u erature and T is the time-dependent factor in u.) Substitute (3.2) into (3.1); we

$$T\nabla^2 F = \frac{1}{\alpha^2} F \frac{dT}{dt}.$$

Next divide (3.3) by FT to get

(3.4) $$\frac{1}{F}\nabla^2 F = \frac{1}{\alpha^2}\frac{1}{T}\frac{dT}{dt}.$$

The left side of this identity is a function only of the space variables x, y, z, and the right side is a function only of time. Therefore both sides are the same constant and we can write

(3.5) $$\frac{1}{F}\nabla^2 F = -k^2 \quad \text{or} \quad \nabla^2 F + k^2 F = 0 \quad \text{and}$$
$$\frac{1}{\alpha^2}\frac{1}{T}\frac{dT}{dt} = -k^2 \quad \text{or} \quad \frac{dT}{dt} = -k^2\alpha^2 T.$$

The time equation can be integrated to give

(3.6) $$T = e^{-k^2\alpha^2 t}.$$

We can see a physical reason here for choosing the separation constant $(-k^2)$ to be negative. As t increases, the temperature of a body might decrease to zero as in (3.6), but it could not increase to infinity as it would if we had used $+k^2$ in (3.5) and (3.6). The space equation in (3.5) is the Helmholtz equation (1.5) as promised. You will find (Problem 10) that the space part of the wave equation is also the Helmholtz equation.

Let us now consider the flow of heat through a slab of thickness l (for example, the wall of a refrigerator). We shall assume that the faces of the slab are so large that we may neglect any end effects and assume that heat flows only in the x direction (Figure 3.1). This problem is then identical with the problem of heat flow in a bar of length l with insulated sides, because in both cases the heat flow is just in the x direction. Suppose the slab has initially a steady-state temperature distribution with the $x = 0$ wall at 0° and the $x = l$ wall at 100°. From $t = 0$ on, let the $x = l$ wall (as well as the $x = 0$ wall) be held at 0°. We want to find the temperature at any x (in the slab) at any later time.

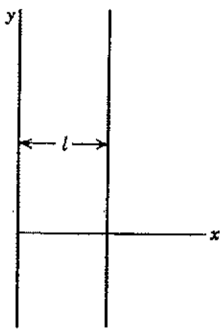


FIGURE 3.1

First, we find the initial steady-state temperature distribution. You probably already know that this is linear, but it is interesting to see this from our equations. The initial steady-state temperature u_0 satisfies Laplace's equation, which in this one-dimensional case is $d^2u_0/dx^2 = 0$. The solution of this equation is $u_0 = ax + b$, where a and b are constants which must be found to fit the given conditions. Since $u_0 = 0$ at $x = 0$ and $u_0 = 100$ at $x = l$, we have

(3.7) $$u_0 = \frac{100}{l}x.$$

From $t = 0$ on, u satisfies the heat flow equation (3.1). We have already separated this; the solutions are (3.2) where $T(t)$ is given by (3.6) and $F(x)$ satisfies the first of equations (3.5), namely

(3.8) $$\nabla^2 F + k^2 F = 0 \quad \text{or} \quad \frac{d^2F}{dx^2} + k^2 F = 0.$$

r this one-dimensional problem, F is a function only of x.) The solutions of (3.8) are

9) $$F(x) = \begin{cases} \sin kx, \\ \cos kx, \end{cases}$$

l the basic solutions (3.2) are

10) $$u = \begin{cases} e^{-k^2\alpha^2 t} \sin kx, \\ e^{-k^2\alpha^2 t} \cos kx. \end{cases}$$

discard the cos kx solution for this problem because we are given $u = 0$ at $x = 0$. o we want $u = 0$ at $x = l$; this will be true if $\sin kl = 0$, that is, $kl = n\pi$, or $k = n\pi/l$ envalues). Our basic solutions (or eigenfunctions) are then

1) $$u = e^{-(n\pi\alpha/l)^2 t} \sin \frac{n\pi x}{l}$$

the solution of our problem will be the series

2) $$u = \sum_{n=1}^{\infty} b_n e^{-(n\pi\alpha/l)^2 t} \sin \frac{n\pi x}{l}.$$

$= 0$, we want $u = u_0$ as in (3.7), that is,

3) $$u = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} = u_0 = \frac{100}{l} x.$$

means finding the Fourier sine series for $(100/l)x$ on $(0, l)$; the result (from lem 1) for the coefficients is

4) $$b_n = \frac{100}{l} \frac{2l}{\pi} \frac{1}{n} (-1)^{n-1} = \frac{200}{\pi} \frac{(-1)^{n-1}}{n}.$$

n we get the final solution by substituting (3.14) into (3.12); this gives

) $$u = \frac{200}{\pi}\left[e^{-(\pi\alpha/l)^2 t} \sin \frac{\pi x}{l} - \tfrac{1}{2} e^{-(2\pi\alpha/l)^2 t} \sin \frac{2\pi x}{l} + \tfrac{1}{3} e^{-(3\pi\alpha/l)^2 t} \sin \frac{3\pi x}{l} + \cdots\right].$$

e can now do some variations of this problem. Suppose the final temperatures of aces are given as two different constant values different from zero. Then, as for the l steady state, the final steady state is a linear function of distance. The series) tends to a final steady state of zero; to obtain a solution tending to some other steady state, we add to (3.12) the linear function u_f representing the correct final y state. Thus we write instead of (3.12)

) $$u = \sum_{n=1}^{\infty} b_n e^{-(n\pi\alpha/l)^2 t} \sin \frac{n\pi x}{l} + u_f.$$

Then for $t = 0$, the equation corresponding to (3.13) is

(3.17) $$u_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} + u_f$$

or

(3.18) $$u_0 - u_f = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}.$$

Thus when $u_f \neq 0$, it is $u_0 - u_f$ rather than u_0 which must be expanded in a Fourier series.

So far we have had the boundary temperatures given. We could, instead, have the faces insulated; then no heat flows in or out of the body. This will be true if the normal derivative $\partial u/\partial n$ (see Problem 2.14) of the temperature is zero at the boundary. (When the boundary values of u are given, the problem is called a *Dirichlet problem*; when the boundary values of the normal derivative $\partial u/\partial n$ are given, the problem is called a *Neumann problem.*) For the one-dimensional case we have considered, we replace the condition $u = 0$ at $x = 0$ and l by the condition $\partial u/\partial x = 0$ at $x = 0$ and l if the faces are insulated. This means that the useful basic solution in (3.10) is now the one containing $\cos kx$; note carefully that we must include the constant term (corresponding to $k = 0$). See Problem 7.

PROBLEMS, SECTION 3

1. Verify the coefficients in equation (3.14).

2. A bar 10 cm long with insulated sides is initially at 100°. Starting at $t = 0$, the ends are held at 0°. Find the temperature distribution in the bar at time t.

   Answer: $u = \dfrac{400}{\pi} \displaystyle\sum_{\text{odd } n} \frac{1}{n} e^{-(n\pi\alpha/10)^2 t} \sin \frac{n\pi x}{10}.$

3. In the initial steady state of an infinite slab of thickness l, the face $x = 0$ is at 0° and the face $x = l$ is at 100°. From $t = 0$ on, the $x = 0$ face is held at 100° and the $x = l$ face at 0°. Find the temperature distribution at time t.

   Answer: $u = 100 - \dfrac{100x}{l} - \dfrac{400}{\pi} \displaystyle\sum_{\text{even } n} \frac{1}{n} e^{-(n\pi\alpha/l)^2 t} \sin \frac{n\pi x}{l}.$

4. At $t = 0$, two flat slabs each 5 cm thick, one at 0° and one at 20°, are stacked together, and then the surfaces are kept at 0°. Find the temperature as a function of x and t for $t > 0$.

5. Two slabs, each 1 inch thick, each have one surface at 0° and the other surface at 100°. At $t = 0$, they are stacked with their 100° faces together and then the outside surfaces are held at 100°. Find $u(x, t)$ for $t > 0$.

6. Show that the following problem is easily solved using (3.15): The ends of a bar are initially at 20° and 150°; at $t = 0$ the 150° end is changed to 50°. Find the time-dependent temperature distribution.

7. A bar of length l with insulated sides has its ends also insulated from time $t = 0$ on. Initially the temperature is $u = x$, where x is the distance from one end. Determine the

temperature distribution inside the bar at time t. *Hints and comments:* See the last paragraph of this section, above, and also Problem 2.14. Show that the $k = 0$ solutions are x and constant (time independent). Note that here (unlike Problem 2.15) you do not need the extra solution (namely x) for $k = 0$ since the final steady state is a constant and this is included in the solutions (3.10). Also note that we *did* need the $k = 0$ solutions in the discussion following (3.15) but were able to simplify the work by observing that these linear solutions simply give the final steady state.

Answer: $u = \frac{l}{2} - \frac{4l}{\pi^2} \sum_{\text{odd } n} \frac{1}{n^2} \cos \frac{n\pi x}{l} e^{-(n\pi\alpha/l)^2 t}$.

8. A bar of length 2 is initially at 0°. From $t = 0$ on, the $x = 0$ end is held at 0° and the $x = 2$ end at 100°. Find the time-dependent temperature distribution.

9. Solve Problem 8 if, for $t > 0$, the $x = 0$ end of the bar is insulated and the $x = 2$ end is held at 100°. See Problem 7 above and Problem 6.8 of Chapter 12.

10. Separate the wave equation (1.4) into a space equation and a time equation as we did the heat flow equation, and show that the space equation is the Helmholtz equation for this case also.

4. THE WAVE EQUATION; THE VIBRATING STRING

Let a string (for example, a piano or violin string) be stretched tightly and its ends fastened to supports at $x = 0$ and $x = l$. When the string is vibrating, its vertical displacement y from its equilibrium position along the x axis depends on x and t. We assume that the displacement y is always very small and that the slope $\partial y/\partial x$ of the string at any point at any time is small. In other words, we assume that the string never gets very far away from its stretched equilibrium position; in fact, we do not distinguish between the length of the string and the distance between the supports, although it is clear that the string must stretch a little as it vibrates out of its equilibrium position. Under these assumptions, the displacement y satisfies the (one-dimensional) wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}. \tag{4.1}$$

The constant v depends on the tension and the linear density of the string; it is called the wave velocity because it is the velocity with which a disturbance at one point of the string would travel along the string. To separate the variables, we substitute

$$y = X(x)T(t) \tag{4.2}$$

into (4.1), and get (Problem 3.10)

$$\frac{1}{X} \frac{d^2 X}{dx^2} = \frac{1}{v^2} \frac{1}{T} \frac{d^2 T}{dt^2} = -k^2,$$

or

$$X'' + k^2 X = 0, \quad \ddot{T} + k^2 v^2 T = 0. \tag{4.3}$$

We can see from the physical problem why we use a negative separation constan here; the solutions are to describe vibrations which are represented by sines an cosines, not by real exponentials. Of course, if we tried taking $+k^2$, we would als discover mathematically that we could not satisfy the boundary conditions for real k.

Recall the following notation used in discussing wave phenomena (see Chapter : Problem 2.17):

ν = frequency (sec^{-1}) $\quad \omega = 2\pi\nu$ = angular frequency (radians)
λ = wavelength $\quad k = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{v} = \frac{\omega}{v}$ = wave number
$v = \lambda\nu$

The solutions of the two equations in (4.3) are

$$X = \begin{cases} \sin kx, \\ \cos kx, \end{cases} \qquad T = \begin{cases} \sin kvt = \sin \omega t, \\ \cos kvt = \cos \omega t. \end{cases} \tag{4.4}$$

The basic solutions (4.2) for y are

$$y = \begin{cases} \sin kx \sin \omega t, \\ \sin kx \cos \omega t, \\ \cos kx \sin \omega t, \\ \cos kx \cos \omega t, \end{cases} \qquad \text{where } \omega = kv. \tag{4.5}$$

Since the string is fastened at $x = 0$ and $x = l$, we must have $y = 0$ for these values (x and all t. This means that we want only the $\sin kx$ terms in (4.5), and also we select so that $\sin kl = 0$, or $k = n\pi/l$. The solutions (4.5) then become

$$y = \begin{cases} \sin \frac{n\pi x}{l} \sin \frac{n\pi vt}{l}, \\ \sin \frac{n\pi x}{l} \cos \frac{n\pi vt}{l}. \end{cases} \tag{4.6}$$

The particular combination of solutions (4.6) that we should take to solve a give problem depends on the initial conditions. For example, suppose the string is starte vibrating by plucking (that is, pulling it aside a small distance h at the center an letting go). Then we are given the shape of the string at $t = 0$, namely $y_0 = f(x)$ as i Figure 4.1, and also the fact that the velocity $\partial y/\partial t$ of points on the string is zero : $t = 0$. (Do not confuse $\partial y/\partial t$ with the wave velocity v; there is no relation betwee

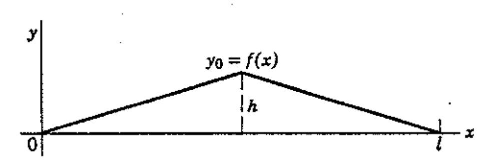


FIGURE 4.1

them.) In (4.6), we must then discard the term containing $\sin(n\pi vt/l)$ since its time derivative is not zero when $t = 0$. We then write the solution for this problem in the form

$$y = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi vt}{l}. \tag{4.7}$$

The coefficients b_n are to be determined so that at $t = 0$ we have $y_0 = f(x)$, that is,

$$y_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} = f(x). \tag{4.8}$$

As in previous problems, we find the coefficients in the Fourier sine series for the given $f(x)$ and substitute them into (4.7). The result is (Problem 1)

$$y = \frac{8h}{\pi^2}\left(\sin \frac{\pi x}{l} \cos \frac{\pi vt}{l} - \frac{1}{9} \sin \frac{3\pi x}{l} \cos \frac{3\pi vt}{l} + \cdots\right). \tag{4.9}$$

Another way to start the string vibrating is to hit it (a piano string, for example). In this case the initial conditions would be $y = 0$ at $t = 0$, and the velocity $\partial y/\partial t$ at $t = 0$ given as a function of x (that is, the velocity of each point of the string is given at $t = 0$). This time we discard in (4.6) the term containing $\cos (n\pi vt/l)$ because it is not zero at $t = 0$. The solution of the problem is then of the form

$$y = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \sin \frac{n\pi vt}{l}. \tag{4.10}$$

Here the coefficients must be determined so that

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} B_n \frac{n\pi v}{l} \sin \frac{n\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} = V(x), \tag{4.11}$$

that is, $V(x)$, the given initial velocity, must be expanded in a Fourier sine series (see Problems 5 to 8).

Suppose the string is vibrating in such a way that instead of an infinite series for y, we have just one of the solutions (4.6), say

$$y = \sin \frac{n\pi x}{l} \sin \frac{n\pi vt}{l} \tag{4.12}$$

for some one value of n. The largest value of $\sin (n\pi vt/l)$, for any t, is 1, and the shape of the string then is

$$y = \sin \frac{n\pi x}{l}. \tag{4.13}$$

Graphs of (4.13) are sketched in Figure 4.2 for $n = 1, 2, 3, 4$. (The graphs are exaggerated! Remember that the displacements are actually very small.) Consider a point x on the string; for this point $\sin (n\pi x/l)$ is some number, say A. Then the displacement of this point at time t is (from (4.12))

$$y = A \sin \frac{n\pi vt}{l}. \tag{4.14}$$

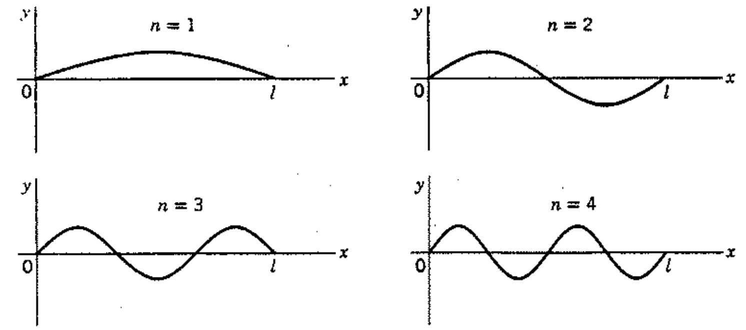
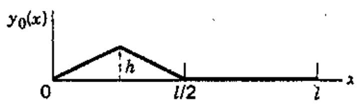


FIGURE 4.2

As time passes, this point of the string oscillates up and down with frequency ν_n given by $\omega_n = n\pi v/l = 2\pi\nu_n$ or $\nu_n = nv/(2l)$; the amplitude of the oscillation at this point is $A = \sin (n\pi x/l)$ (see Figure 4.2). Other points of the string oscillate with different amplitudes but the *same* frequency. This is the frequency of the musical note which the string is producing. If $n = 1$ (see Figure 4.2), the frequency is $v/(2l)$; in music this tone is called the fundamental or first harmonic. If $n = 2$, the frequency is just twice that of the fundamental; this tone is called the first overtone or the second harmonic; etc. All the frequencies which this string can produce are multiples of the fundamental. These frequencies are called the *characteristic frequencies* of the string. (They are proportional to the *characteristic values* or *eigenvalues*, $k = n\pi/l$.) The corresponding ways in which the string may vibrate producing a pure tone of just one frequency [that is, with y given by (4.12) for one value of n)] are called the *normal modes of vibration.* The first four normal modes are indicated in Figure 4.2. Any vibration is a combination of these normal modes [for example, (4.9) or (4.10)]. The solution (4.12) (for *one n*) describing one normal mode, is a *characteristic function* or *eigenfunction.*

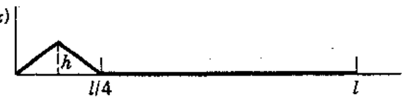
PROBLEMS, SECTION 4

1. Complete the plucked string problem to get equation (4.9).
2. A string of length l has a zero initial velocity and a displacement $y_0(x)$ as shown. (This initial displacement might be caused by stopping the string at the center and plucking half of it.) Find the displacement as a function of x and t.

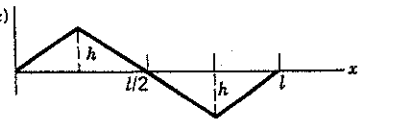


Answer: $y = \frac{8h}{\pi^2} \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{l} \cos \frac{n\pi vt}{l}$, where $B_n = (2 \sin n\pi/4 - \sin n\pi/2)/n^2$.

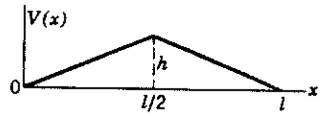
3. Solve Problem 2 if the initial displacement is: $y_0(x)$



4. Solve Problem 2 if the initial displacement is: $y_0(x)$

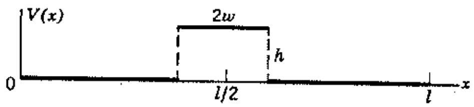


5. A string of length l is initially stretched straight; its ends are fixed for all t. At time $t = 0$, its points are given the *velocity* $V(x) = (\partial y/\partial t)_{t=0}$ as indicated in the diagram (for example, by hitting the string). Determine the shape of the string at time t, that is, find the displacement y as a function of x and t in the form of a series similar to (4.9). *Warning:* What functions do you need here?



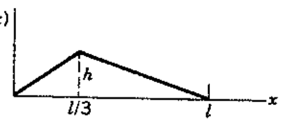
Answer: $y = \dfrac{8hl}{\pi^3 v}\left(\sin\dfrac{\pi x}{l}\sin\dfrac{\pi vt}{l} - \dfrac{1}{3^3}\sin\dfrac{3\pi x}{l}\sin\dfrac{3\pi vt}{l} + \dfrac{1}{5^3}\sin\dfrac{5\pi x}{l}\sin\dfrac{5\pi vt}{l} - \cdots\right).$

Do Problem 5 if the initial velocity $V(x) = (\partial y/\partial t)_{t=0}$ is as shown.



Answer: $y = \dfrac{4hl}{\pi^2 v}\left(\sin\dfrac{\pi w}{l}\sin\dfrac{\pi x}{l}\sin\dfrac{\pi vt}{l} - \dfrac{1}{9}\sin\dfrac{3\pi w}{l}\sin\dfrac{3\pi x}{l}\sin\dfrac{3\pi vt}{l} + \cdots\right).$

Solve Problem 5 if the initial velocity is:



Solve Problem 5 if the initial velocity is

$$V(x) = \begin{cases} \sin 2\pi x/l & \text{for } 0 < x < l/2, \\ 0 & \text{for } l/2 < x < l. \end{cases}$$

In each of Problems 1 to 8, find the frequency of the most important harmonic.

STEADY-STATE TEMPERATURE IN A CYLINDER

onsider the following problem. Find the steady-state mperature distribution u in a semi-infinite solid cylinder igure 5.1) of radius 1 if the base is held at 100° and the rved sides at 0°. This sounds very much like the prob- n of the temperature distribution in a semi-infinite ate. However, it is not convenient here to use the solu- ns in rectangular coordinates, because the boundary ndition $u = 0$ is given for $r = 1$ rather than for constant ues of x or y. The natural variables for the problem are cylindrical coordinates r, θ, z. The temperature u side the cylinder satisfies Laplace's equation since there no sources of heat there.

Laplace's equation in cylindrical coordinates is (see apter 10, Section 9)

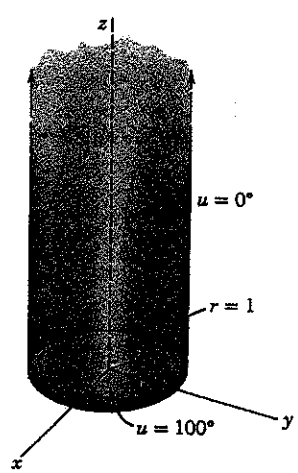


FIGURE 5.1

$$\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{1}$$

To separate the variables, we assume a solution of the form

$$u = R(r)\Theta(\theta)Z(z). \tag{5.2}$$

Substitute (5.2) into (5.1) and divide by $R\Theta Z$ to get

$$\frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{r^2}\frac{d^2\Theta}{d\theta^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0. \tag{5.3}$$

The last term is a function only of z, while the other two terms do not contain z. Therefore the last term is a constant and the *sum* of the first two terms is minus the same constant. Notice that neither of the first two terms is constant alone since both contain r.

In order to say that a term is constant, we must be sure that
(a) it is a function of only one variable, and
(b) that variable does not appear elsewhere in the equation.

Thus we have

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = k^2, \qquad Z = \begin{cases} e^{kz}, \\ e^{-kz}. \end{cases} \tag{5.4}$$

Since we want the temperature u to tend to zero as z tends to infinity, we call the separation constant $+k^2$ $(k > 0)$ and then use only the e^{-kz} solution. Next write (5.3) with the last term replaced by k^2—see (5.4).

$$\frac{1}{R}\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{r^2}\frac{d^2\Theta}{d\theta^2} + k^2 = 0.$$

We can separate the variables by multiplying by r^2.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} + k^2r^2 = 0. \tag{5.5}$$

In (5.5) the second term is a function of θ only, and the other terms are independent of θ. Thus we have

$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -n^2, \qquad \Theta = \begin{cases} \sin n\theta, \\ \cos n\theta. \end{cases} \tag{5.6}$$

Here we must use $-n^2$ as the separation constant and then require n to be an integer for the following reason. When we locate a point using polar coordinates, we can choose the angle as θ or as $\theta + 2m\pi$ where m is any integer. But regardless of the value of m, there is one physical point and one temperature there. The mathematical formula for the temperature at the point must give the same value at θ as at $\theta + 2m\pi$, that is, the temperature must be a periodic function of θ with period 2π. This is true only if the Θ solutions are sines and cosines instead of exponentials (hence the negative separation constant) and the constant n is an integer (to give period 2π).

Finally, the r equation is

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) - n^2 + k^2r^2 = 0$$

7) $$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (k^2r^2 - n^2)R = 0.$$

is is a Bessel equation with solutions $J_n(kr)$ and $N_n(kr)$ [see Chapter 12, equation .2); put $x = r$, and $a = k$]. Since the base of the cylinder we are considering tains the origin, we can use only the J_n and not the N_n solutions since N_n becomes nite at the origin. Hence we have

3) $$R(r) = J_n(kr).$$

can determine the possible values of k from the condition that $u = 0$ on the curved face of the cylinder, that is, $u = 0$ when $r = 1$ (for all θ and z) or $R(r) = 0$ for 1. From (5.8), we have

) $$R_{r=1} = J_n(k) = 0.$$

is the possible values of k are the zeros of J_n. The basic solutions for u are then

10) $$u = \begin{cases} J_n(kr) \sin n\theta \, e^{-kz}, \\ J_n(kr) \cos n\theta \, e^{-kz}, \end{cases}$$

ere k is a zero of J_n.

or our problem, the base of the cylinder is held at a constant temperature of 100°. e turn the cylinder through any angle the boundary conditions are not changed; the solution does not depend on the angle θ. This means that we use $\cos n\theta$ with 0 in (5.10). The possible values of k are the zeros of J_0; call these zeros k_m, where 1, 2, 3, $\cdots$. Then there are an infinite number of solutions of the form (5.10) (one esponding to each zero of J_0), and we shall write the solution of our problem as a s of such solutions (eigenfunctions):

) $$u = \sum_{m=1}^{\infty} c_m J_0(k_m r) e^{-k_m z}.$$

n $z = 0$, we want $u = 100$, that is,

) $$u_{z=0} = \sum_{m=1}^{\infty} c_m J_0(k_m r) = 100.$$

should remind you of Fourier series; here we want to expand 100 in a series of l functions instead of a series of sines or cosines. We proved (see Chapter 12, on 19) that the functions $J_0(k_m r)$ are orthogonal on (0, 1) with respect to the t function r. We can then find the coefficients c_m in (5.12) by the same method in finding the coefficients in a Fourier sine or cosine series. (For this reason, series

like (5.12) are often called Fourier-Bessel series.) Multiply (5.12) by $rJ_0(k_\mu r)$, $\mu = 1, 2, 3, \cdots$, and integrate term by term from $r = 0$ to $r = 1$. Because of the orthogonality [see Chapter 12, equation (19.10)], all terms of the series drop out except the term with $m = \mu$, and we have

(5.13) $$c_\mu \int_0^1 r[J_0(k_\mu r)]^2 \, dr = \int_0^1 100 r J_0(k_\mu r) \, dr.$$

For each value of $\mu = 1, 2, 3, \cdots$, (5.13) gives one of the coefficients in (5.11) and (5.12); thus any c_m in (5.11) is given by (5.13) with μ replaced by m.

We need to evaluate the integrals in (5.13). Equation (19.10) of Chapter 12 gives

(5.14) $$\int_0^1 r[J_0(k_m r)]^2 \, dr = \tfrac{1}{2}J_1^2(k_m).$$

By equation (15.1) of Chapter 12

$$\frac{d}{dx}[xJ_1(x)] = xJ_0(x).$$

If we put $x = k_m r$ in this formula, we get

$$\frac{1}{k_m}\frac{d}{dr}[k_m r J_1(k_m r)] = k_m r J_0(k_m r).$$

Canceling one k_m factor and integrating, we have

(5.15) $$\int_0^1 rJ_0(k_m r) \, dr = \frac{1}{k_m} rJ_1(k_m r)\Big|_0^1 = \frac{1}{k_m} J_1(k_m).$$

Now we write (5.13) for c_m, substitute the values of the integrals from (5.14) and (5.15), and solve for c_m. The result is

(5.16) $$c_m = \frac{100 J_1(k_m)}{k_m} \cdot \frac{2}{J_1^2(k_m)} = \frac{200}{k_m J_1(k_m)}.$$

Warning: Remember that k_m is a zero of J_0, not of J_1. In some tables you may find tabulated the values of J_1 (or of $J_0' = -J_1$) at the zeros of J_0; if not, you can first find the values of k_m (zeros of J_0) and then interpolate in a J_1 table to find the values of $J_1(k_m)$. With the c's given by (5.16), (5.11) is the solution of our problem. The numerical value of the temperature at any point can be found by computing a few terms of the series (Problem 1).

Suppose the given temperature of the base of the cylinder is more complicated than just a constant value, say $f(r, \theta)$, some function of r and θ. Down to (5.10) we proceed as before. But now the series solution is more complicated than (5.11) since we must include all J_n's instead of just J_0. We need a double subscript on the numbers k which are the zeros of the Bessel functions; by k_{mn} we shall mean the mth positive zero of J_n, where $n = 0, 1, 2, \cdots$ and $m = 1, 2, 3, \cdots$. The temperature u is a double infinite series, summed over the indices m, n of all zeros of all the J_n's:

(5.17) $$u = \sum \sum J_n(k_{mn} r)(a_{mn} \cos n\theta + b_{mn} \sin n\theta) e^{-k_{mn} z}.$$

t $z = 0$, we want $u = f(r, \theta)$. Thus we write

$$u_{z=0} = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} J_n(k_{mn} r)(a_{mn} \cos n\theta + b_{mn} \sin n\theta) = f(r, \theta). \quad \text{.18)}$$

determine the coefficients a_{mn}, multiply this equation by $J_\nu(k_{\mu\nu} r) \cos \nu\theta$ and inte- ate over the whole base of the cylinder (0 to 2π for θ, 0 to 1 for r). Because of the hogonality of the functions $\sin n\theta$ and $\cos n\theta$ on $(0, 2\pi)$, all the b_{mn} terms drop out, d only the a_{mn} terms for $n = \nu$ remain. Because of the orthogonality of the functions $k_{mn} r)$ (one n, all m), only the one term $a_{\mu\nu}$ remains. Thus we have

$$\int_0^1 \int_0^{2\pi} f(r, \theta) J_\nu(k_{\mu\nu} r) \cos \nu\theta \, r \, dr \, d\theta = a_{\mu\nu} \int_0^1 \int_0^{2\pi} J_\nu^2(k_{\mu\nu} r) \cos^2 \nu\theta \, r \, dr \, d\theta \quad \text{19)}$$

$$= a_{\mu\nu} \cdot \tfrac{1}{2} J_{\nu+1}^2(k_{\mu\nu}) \cdot \pi.$$

he r integral is given by (19.10) of Chapter 12, and the θ integral by Chapter 7, tion 4.] Notice how the weight function r in the Bessel function integral arises here part of the element of area in polar coordinates. Similarly, we can find

$$b_{\mu\nu} = \frac{2}{\pi J_{\nu+1}^2(k_{\mu\nu})} \int_0^1 \int_0^{2\pi} f(r, \theta) J_\nu(k_{\mu\nu} r) \sin \nu\theta \, r \, dr \, d\theta. \quad \text{0)}$$

substituting the values of the a and b coefficients from (5.19) and (5.20) into (5.17), find the solution to the problem.

BLEMS, SECTION 5

Compute numerically the coefficients (5.16) of the first three terms of the series (5.11) for the steady-state temperature in a solid semi-infinite cylinder when $u = 0$ at $r = 1$, and $u = 100$ at $z = 0$. Find u at $r = \frac{1}{2}$, $z = 1$.

Find the steady-state temperature distribution in a solid semi-infinite cylinder if the boundary temperatures are $u = 0$ at $r = 1$ and $u = y = r \sin \theta$ at $z = 0$. *Hints:* In (5.10) you want the solution containing $\sin \theta$; therefore you want the functions J_1. You will need to integrate $r^2 J_1$; follow the text method of integrating rJ_0 just before (5.15).

Answer: $u = \sum_{m=1}^{\infty} \frac{2}{k_m J_2(k_m)} J_1(k_m r) e^{-k_m z} \sin \theta, \qquad k_m = \text{zeros of } J_1.$

Find the steady-state temperature distribution in a solid cylinder of height 10 and radius 1 if the top and the curved surface are held at 0° and the base at 100°.

A flat circular plate of radius 1 is initially at temperature 100°. From time $t = 0$ on, the circumference of the plate is held at 0°. Find the time-dependent temperature distribution $u(r, \theta, t)$. *Hint:* Separate variables in equation (3.1) in polar coordinates.

Do Problem 4 if the initial temperature distribution is $u(r, \theta, t = 0) = 100r \sin \theta$.

Consider Problem 4 if the initial temperature distribution is given as some function $f(r, \theta)$. The solution is, in general, a double infinite series similar to (5.17). Find formulas for the coefficients in the series.

7. Find the steady-state temperature distribution in a solid cylinder of height 20 and radius 3 if the flat ends are held at 0° and the curved surface at 100°. *Hints:* Use $-k^2$ in (5.4). Also see Chapter 12, Sections 17 and 20.

8. Water at 100° is flowing through a long pipe of radius 1 rapidly enough so that we may assume that the temperature is 100° at all points. At $t = 0$, the water is turned off and the surface of the pipe is maintained at 40° from then on (neglect the wall thickness of the pipe). Find the temperature distribution in the water as a function of r and t. Note that you need only consider a cross section of the pipe.

   Answer: $u = 40 + \sum_{m=1}^{\infty} \frac{120}{k_m J_1(k_m)} J_0(k_m r) e^{-(\alpha k_m)^2 t}, \qquad \text{where} \quad J_0(k_m) = 0.$

9. Find the steady-state distribution of temperature in a cube of side 10 if the temperature is 100° on the face $z = 0$ and 0° on the other five faces. *Hint:* Separate Laplace's equation in three dimensions in rectangular coordinates, and follow the methods of Section 2. You will want to expand 100 in the double Fourier series

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin \frac{n\pi x}{l} \sin \frac{m\pi y}{l}.$$

   The coefficients a_{nm} are determined by making use of the orthogonality of the functions $\sin (n\pi x/l) \sin (m\pi y/l)$ over the square, that is,

$$\int_0^l \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi y}{l} \sin \frac{p\pi x}{l} \sin \frac{q\pi y}{l} \, dx \, dy = 0 \qquad \text{unless} \quad \begin{cases} n = p, \\ m = q. \end{cases}$$

10. A cube is originally at 100°. From $t = 0$ on, the faces are held at 0°. Find the time-dependent temperature distribution. *Hint:* This problem leads to a triple Fourier series; see the double Fourier series in Problem 9 and generalize it to three dimensions.

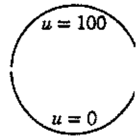
11. The following two $R(r)$ equations arise in various separation of variables problems in polar, cylindrical, or spherical coordinates:

$$r \frac{d}{dr}\left(r \frac{dR}{dr}\right) = n^2 R,$$

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = l(l+1)R.$$

   There are various ways of solving them: They are a standard kind of equation (often called Euler or Cauchy equations—see Chapter 8, Section 7d); you could use power series methods; given the fact that the solutions are just powers of r, it is easy to find the powers. Choose any method you like, and solve the two equations for future reference. Consider the case $n = 0$ separately. Is this necessary for $l = 0$?

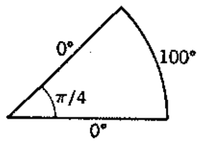
12. Separate Laplace's equation in two dimensions in polar coordinates and solve the r and θ equations. (See Problem 11.) Remember that for the θ equation, only periodic solutions are of interest. Use your results to solve the problem of the steady-state temperature in a circular plate if the upper semicircular boundary is held at 100° and the lower at 0°. (Continued, p. 564.)



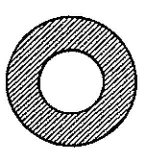
Comment: Another physical problem whose mathematical solution is identical with this temperature problem is this: Find the electrostatic potential inside a capacitor formed by two half-cylinders, insulated from each other and maintained at potentials 0 and 100.

Answer: $u = 50 + \frac{200}{\pi} \sum_{\text{odd } n} \left(\frac{r}{a}\right)^n \frac{\sin n\theta}{n}$.

13. Find the steady-state distribution of temperature in the sector of a circular plate of radius 10 and angle $\pi/4$ if the temperature is maintained at 0° along the radii and at 100° along the curved edge. *Hint:* See Problem 12.



14. Find the steady-state temperature distribution in a circular annulus (shaded area) of inner radius 1 and outer radius 2 if the inner circle is held at 0° and the outer circle has half its circumference at 0° and half at 100°. *Hint:* Don't forget the r solutions corresponding to $k = 0$.



15. Solve Problem 14 if the temperatures of the two circles are interchanged.

6. VIBRATION OF A CIRCULAR MEMBRANE

A circular membrane (for example, a drumhead) is attached to a rigid support along its circumference. Find the characteristic vibration frequencies and the corresponding normal modes of vibration.

Take the (x, y) plane to be the plane of the circular support and take the origin at its enter. Let $z(x, y, t)$ be the displacement of the membrane from the (x, y) plane. Then satisfies the wave equation

$$\nabla^2 z = \frac{1}{v^2}\frac{\partial^2 z}{\partial t^2}. \tag{6.1}$$

utting

$$z = F(x, y)T(t), \tag{6.2}$$

e separate (6.1) into a space equation (Helmholtz) and a time equation (see Problem 10 and Section 3). We get the two equations

$$\nabla^2 F + k^2 F = 0 \quad \text{and} \quad \ddot{T} + k^2 v^2 T = 0. \tag{6.3}$$

ecause the membrane is circular we write ∇^2 in polar coordinates (see Chapter 10, ection 9); then the F equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} + k^2 F = 0. \tag{6.4}$$

/hen we put

$$F = R(r)\Theta(\theta), \tag{6.5}$$

(6.4) becomes (5.5), and the separated equations and their solutions are just (5.6), (5.7), and (5.8). The time equation in (6.3) is the same as in (4.3) and the solutions for T are the same as in (4.4). The basic solutions for z are then

$$z = J_n(kr)\begin{cases}\sin n\theta \sin kvt\\ \sin n\theta \cos kvt\\ \cos n\theta \sin kvt\\ \cos n\theta \cos kvt\end{cases} \tag{6.6}$$

Just as in Section 5, n must be an integer. To find possible values of k, we use the fac that the membrane is attached to a rigid frame at $r = 1$, so we must have $z = 0$ a $r = 1$ for all values of θ and t. Thus $J_n(k) = 0$, and we see that the possible value (eigenvalues) of k for each J_n are k_{mn}, the zeros of J_n. For a given initial displacemen or velocity of the membrane, we could find z as a double series as we found (5.17) i the cylinder temperature problem. However, here we shall do something different namely investigate the separate normal modes of vibration and their frequencies.

Recall that for the vibrating string (Section 4), each n gives a different frequency an a corresponding normal mode of vibration (Figure 4.2). The frequencies are $\nu_n =$ $nv/(2l)$; all frequencies are integral multiples of the frequency $\nu_1 = v/(2l)$ of the funda mental. For the circular membrane, the frequencies are [from (6.6) or (4.4)]

$$\nu = \frac{\omega}{2\pi} = \frac{kv}{2\pi}.$$

The possible values of k are the zeros k_{mn} of the Bessel functions. Each value of k_m gives a frequency $\nu_{mn} = k_{mn}v/(2\pi)$, so we have a doubly infinite set of characteristi frequencies and the corresponding normal modes of vibration. All these frequencies ar different, and they are not integral multiples of the fundamental as is true for th string. This is why a drum is less musical than a violin. Using tables you can look u several k_{mn} values (Problem 2) and find the frequencies as (nonintegral) multiples of th fundamental (which corresponds to k_{10}, the first zero of J_0). Let us sketch a fe graphs (Figure 6.1) of the normal vibration modes corresponding to those in Figure 4. for the string, and write the corresponding formulas (eigenfunctions) for the displace ment z given in (6.6). (For simplicity, we have used just the $\cos n\theta \cos kvt$ solutions i Figure 6.1.) In the fundamental mode of vibration corresponding to k_{10}, the membran vibrates as a whole. In the k_{20} mode, it vibrates in two parts as shown, the + pa vibrating up while the − part vibrates down, and vice versa, with the circle betwee them at rest. We can show that there is such a circle (called a nodal line) and find i radius. Since $k_{20} > k_{10}$, the circle $r = k_{10}/k_{20}$ is a circle of radius less than 1; hence is a circle on the membrane. For this value of r, $J_0(k_{20}r) = J_0(k_{20}k_{10}/k_{20})$ $J_0(k_{10}) = 0$, so points on this circle are not displaced. For the k_{11} mode, $\cos\theta = 0$ f $\theta = \pm\pi/2$ and is positive or negative as shown. Continuing in this way you can sketc any normal mode (Problem 1).

It is difficult experimentally to obtain pure normal modes of a vibrating objec However, a complicated vibration will have nodal lines of some kind and it is easy

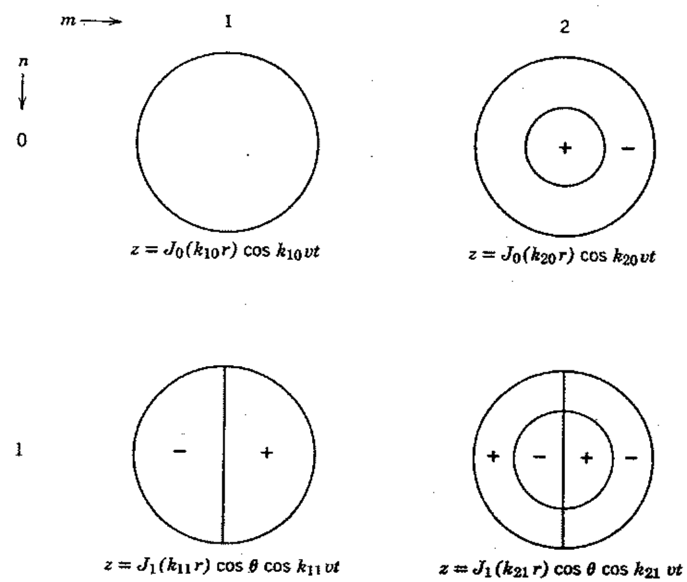


FIGURE 6.1

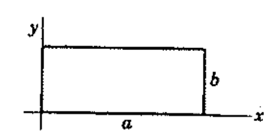
serve these. Fine sand sprinkled on the vibrating object will collect along the nodal es (where there is no vibration) so that you can see them clearly. [For experimental rk on the vibrating circular membrane, see *American Journal of Physics*, Vol. 35 967), p. 1029, and Vol. 40 (1972), p. 186.]

OBLEMS, SECTION 6

Continue Figure 6.1 to show the fundamental modes of vibration of a circular membrane for $n = 0, 1, 2$, and $m = 1, 2, 3$. As in Figure 6.1, write the formula for the displacement z under each sketch.

Look up in tables the first three zeros k_{mn} of each of the Bessel functions J_0, J_1, J_2, and J_3. Find the first six frequencies of a vibrating circular membrane as multiples of the fundamental frequency.

Separate the wave equation in two-dimensional rectangular coordinates x, y. Consider a rectangular membrane as shown, rigidly attached to supports along its sides. Show that its characteristic frequencies are



$$\nu_{nm} = (v/2)\sqrt{(n/a)^2 + (m/b)^2},$$

where n and m are positive integers, and sketch the normal modes of vibration corresponding to the first few frequencies. That is, indicate the nodal lines as we did for the circular membrane in Figure 6.1 and Problem 1.

Next suppose the membrane is square. Show that in this case there may be two or more normal modes of vibration corresponding to a single frequency. (*Hint for one example:* $7^2 + 1^2 = 1^2 + 7^2 = 5^2 + 5^2$.) This is an example of what is called *degeneracy*; we say that there is degeneracy when several different solutions of the wave equation correspond to the same frequency. Sketch several normal modes giving rise to the same frequency.

4. Find the characteristic frequencies for sound vibration in a rectangular box (say a room) of sides a, b, c. *Hint:* Separate the wave equation in three dimensions in rectangular coordinates. This problem is like Problem 3 but for three dimensions instead of two. Discuss degeneracy (see Problem 3).

5. A square membrane of side l is distorted into the shape

$$f(x, y) = xy(l - x)(l - y)$$

and released. Express its shape at subsequent times as an infinite series. *Hint:* Use a double Fourier series as in Problem 5.9.

7. STEADY-STATE TEMPERATURE IN A SPHERE

Find the steady-state temperature inside a sphere of radius 1 when the surface of the upper half is held at 100° and the surface of the lower half at 0°.

Inside the sphere, the temperature u satisfies Laplace's equation. In spherical coordinates this is (see Chapter 10, Section 9)

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0. \tag{7.1}$$

We separate this equation following our standard procedure. Substitute

$$u = R(r)\Theta(\theta)\Phi(\phi) \tag{7.2}$$

into (7.1) and multiply by $r^2/(R\Theta\Phi)$ to get

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{1}{\sin^2\theta}\frac{d^2\Phi}{d\phi^2} = 0. \tag{7.3}$$

If we multiply (7.3) by $\sin^2\theta$, the last term becomes a function of ϕ only and the other two terms do not contain ϕ. Thus we obtain the ϕ equation and its solutions:

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2, \qquad \Phi = \begin{cases}\sin m\phi, \\ \cos m\phi.\end{cases} \tag{7.4}$$

The separation constant must be negative and m an integer to make Φ a periodic function of ϕ [see the discussion after (5.6)].

Equation (7.3) can now be written as

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = 0. \tag{7.5}$$

The first term is a function of r and the last two terms are functions of θ, so we have two equations

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = k, \tag{7.6}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta}\Theta + k\Theta = 0. \tag{7.7}$$

If you compare (7.7) with the equation of Problem 10.2 in Chapter 12, you will see that (7.7) is the equation for the associated Legendre functions if $k = l(l+1)$. Recall that l must be an integer in order for the solution of Legendre's equation to be finite at $x = \cos\theta = \pm 1$, that is, at $\theta = 0$ or π; the same statement is true for the equation for the associated Legendre functions. The corresponding result for (7.7) is that k must be a product of two successive integers; it is then convenient to replace k by $l(l+1)$, where l is an integer. The solutions of (7.7) are then the associated Legendre functions (see Problem 10.2, Chapter 12)

$$\Theta = P_l^m(\cos\theta). \tag{7.8}$$

In (7.6), we put $k = l(l+1)$; you can then easily verify (Problem 5.11) that the solutions of (7.6) are

$$R = \begin{cases} r^l, \\ r^{-l-1}. \end{cases} \tag{7.9}$$

Since we are interested in the interior of the sphere, we discard the solutions r^{-l-1} because they become infinite at the origin. If we were discussing a problem (say about water flow or electrostatic potential) outside the sphere, we would use these solutions and discard the solutions r^l because they become infinite at infinity.

The basic solutions for our problem are then

$$u = r^l P_l^m(\cos\theta)\begin{cases} \sin m\phi \\ \cos m\phi \end{cases} \tag{7.10}$$

[The functions $P_l^m(\cos\theta)\sin m\phi$ and $P_l^m(\cos\theta)\cos m\phi$ are called *spherical harmonics*; also see Problems 16 and 17.] If the surface temperature at $r = 1$ were given as a function of θ and ϕ, we would have a double series (summed on l and m) as in Section 5. For the given surface temperatures in our problem (100° on the top hemisphere and 0° on the lower hemisphere), the temperature is independent of ϕ; thus in (7.10) we must have $m = 0$, $\cos m\phi = 1$. The solutions (7.10) then reduce to $r^l P_l(\cos\theta)$. We write the solution of our problem as a series of such solutions:

$$u = \sum_{l=0}^{\infty} c_l r^l P_l(\cos\theta). \tag{7.11}$$

We determine the coefficients c_l by using the given temperatures when $r = 1$; that is, we must have

$$u_{r=1} = \sum_{l=0}^{\infty} c_l P_l(\cos\theta) = \begin{cases} 100, & 0 < \theta < \dfrac{\pi}{2}, \quad \text{that is,} \quad 0 < \cos\theta < 1, \\ 0, & \dfrac{\pi}{2} < \theta < \pi, \quad \text{that is,} \quad -1 < \cos\theta < 0, \end{cases} \tag{7.12}$$

or

$$u_{r=1} = \sum_{l=0}^{\infty} c_l P_l(x) = 100 f(x), \tag{7.13}$$

where

$$f(x) = \begin{cases} 0, & -1 < x < 0, \\ 1, & 0 < x < 1. \end{cases}$$

(Note that here x just stands for $\cos\theta$ and is not the coordinate x.) In Section 9 of Chapter 12, we expanded this $f(x)$ in a series of Legendre polynomials and obtained:

$$f(x) = \tfrac{1}{2}P_0(x) + \tfrac{3}{4}P_1(x) - \tfrac{7}{16}P_3(x) + \tfrac{11}{32}P_5(x) + \cdots. \tag{7.14}$$

The coefficients c_l in (7.13) are just these coefficients times 100. Substituting the c's into (7.11), we get the final solution:

$$u = 100[\tfrac{1}{2}P_0(\cos\theta) + \tfrac{3}{4}rP_1(\cos\theta) - \tfrac{7}{16}r^3P_3(\cos\theta) + \tfrac{11}{32}r^5P_5(\cos\theta) + \cdots]. \tag{7.15}$$

We can do variations of this problem. Notice that we have not even mentioned so far what temperature scale we are using (Celsius, Fahrenheit, absolute, etc.). This is a very easy adjustment to make once we have a solution in any one scale. To see why, observe that if u is a solution of Laplace's equation $\nabla^2 u = 0$ or of the heat flow equation $\nabla^2 u = (1/\alpha^2)(\partial u/\partial t)$, then $u + C$ and Cu are also solutions for any constant C. If we add, say, 50° to the solution (7.15), we have the temperature distribution inside a sphere with the top half of the surface at 150° and the lower half at 50°. If we multiply the solution (7.15) by 2, we find the temperature distribution with given surface temperatures of 200° and 0°, and so on.

The temperature of the equatorial plane $\theta = \pi/2$ or $\cos\theta = 0$ as given by equations (7.11) to (7.15) is halfway between the top and bottom surface temperatures, because Legendre series, like Fourier series, converge to the midpoint of a jump in the function which was expanded to get the series. To solve the problem of the temperature in a hemisphere given the temperatures of the spherical surface and of the equatorial plane, we need only imagine the lower hemisphere in place and at the proper temperature to give the desired average on the equatorial plane. When the temperature of the equatorial plane is 0°, this amounts to defining the function $f(x)$ in (7.13) on $(-1, 0)$ to make it an odd function.

PROBLEMS, SECTION 7

Find the steady-state temperature distribution inside a sphere of radius 1 when the surface temperatures are as given in Problems 1 to 10.

1. $35(\cos\theta)^4$
2. $\cos\theta - (\cos\theta)^3$
3. $\cos\theta - 3\sin^2\theta$
4. $5\cos^3\theta - 3\sin^2\theta$
5. $|\cos\theta|$
6. $\pi/2 - \theta$. *Hint:* See Chapter 12, Problem 9.4.
7. $\begin{cases} \cos\theta, & 0 < \theta < \pi/2, \quad \text{that is, upper hemisphere,} \\ 0, & \pi/2 < \theta < \pi, \quad \text{that is, lower hemisphere.} \end{cases}$
8. $\begin{cases} 100°, & 0 < \theta < \pi/3, \\ 0°, & \text{otherwise.} \end{cases}$ *Hint:* See Problem 9.8 of Chapter 12.
9. $3\sin\theta\cos\theta\sin\phi$. *Hint:* See equation (7.10) and Chapter 12, equation (10.6).

10. $\sin^2\theta\cos\theta\cos 2\phi - \cos\theta$. (See Problem 9.)

11. Find the steady-state temperature distribution inside a hemisphere if the spherical surface is held at 100° and the equatorial plane at 0°. *Hint:* See the last paragraph of this section above.

12. Do Problem 11 if the curved surface is held at $\cos^2\theta$ and the equatorial plane at zero. *Careful:* The answer does *not* involve P_2; read the last sentence of this section.

13. Find the electrostatic potential outside a conducting sphere of radius a placed in an originally uniform electric field, and maintained at zero potential. *Hint:* Let the original field **E** be in the negative z direction so that $\mathbf{E} = -E_0\mathbf{k}$. Then since $\mathbf{E} = -\nabla\Phi$, where Φ is the potential, we have $\Phi = E_0 z = E_0 r\cos\theta$ (Verify this!) for the original potential. You then want a solution of Laplace's equation $\nabla^2 u = 0$ which is zero at $r = a$ and becomes $u \sim \Phi$ for large r (that is, far away from the sphere). Select the solutions of Laplace's equation in spherical coordinates which have the right θ and ϕ dependence (there are just two such solutions) and find the combination which reduces to zero for $r = a$.

14. Find the steady-state temperature distribution in a spherical shell of inner radius 1 and outer radius 2 if the inner surface is held at 0° and the outer surface has its upper half at 100° and its lower half at 0°. *Hint:* $r = 0$ is not in the region of interest, so the solutions r^{-l-1} in (7.9) should be included. Replace $c_l r^l$ in (7.11) by $(a_l r^l + b_l r^{-l-1})$.

15. A sphere initially at 0° has its surface kept at 100° from $t = 0$ on (for example, a frozen potato in boiling water!). Find the time-dependent temperature distribution. *Hint:* Subtract 100° from all temperatures and solve the problem; then add the 100° to the answer. Can you justify this procedure? Show that the Legendre function required for this problem is P_0 and the r solution is $(1/\sqrt{r})J_{1/2}$ or j_0 [see (17.4) in Chapter 12]. Since spherical Bessel functions can be expressed in terms of elementary functions, the series in this problem can be thought of as either a Bessel series or a Fourier series. Show that the results are identical.

16. Separate the wave equation in spherical coordinates, and show that the θ, ϕ solutions are the spherical harmonics $P_l^m(\cos\theta)e^{\pm im\phi}$ and the r solutions are spherical Bessel functions $j_l(kr)$ and $y_l(kr)$ [Chapter 12, equations (17.4)].

17. The (time-independent) Schrödinger equation in quantum mechanics is

$$\nabla^2\psi + (\epsilon - bV)\psi = 0,$$

where ϵ and b are constants and V is a given function of r, θ, ϕ for each problem. In most simple cases V is a function of r only (no θ, ϕ dependence). (Physically, V is the potential energy, and the fact that it depends only on r implies that we are dealing with central forces, for example, electrostatic or gravitational forces.) Separate the Schrödinger equation in spherical coordinates for the case $V = V(r)$, and show that the θ, ϕ solutions are spherical harmonics (see Problem 16).

POISSON'S EQUATION

e are going to derive Poisson's equation for a simple problem whose answer we know advance. Using our known solution, we shall be able to see a method of solving more fficult problems.

Recall from Chapter 6, Section 8, that the gravitational field is conservative, that is, curl $\mathbf{F} = 0$, and there is a potential function V such that $\mathbf{F} = -\nabla V$. If we consider the gravitational field at a point P due to a point mass m a distance r away, we have

$$V = -\frac{Gm}{r} \quad \text{and} \quad \mathbf{F} = -\frac{Gm}{r^2}\mathbf{u} \tag{8.1}$$

where **u** is a unit vector along r toward P. It is straightforward to show that div $\mathbf{F} = 0$ and V satisfies Laplace's equation (Problem 1), that is,

$$\nabla\cdot\mathbf{F} = -\nabla\cdot\nabla V = -\nabla^2 V = 0. \tag{8.2}$$

Now suppose there are many masses m_i at distances r_i from P. The total potential at P is the sum of the potentials due to the individual m_i, that is,

$$V = \sum_i V_i = -\sum_i \frac{Gm_i}{r_i}$$

and the total gravitational field at P is the vector sum of the fields $\mathbf{F}_i$, that is,

$$\mathbf{F} = -\sum_i \nabla V_i = -\nabla V.$$

Note that we are taking it for granted that none of the masses m_i are *at* P, that is, that no r_i is zero. Since

$$\nabla\cdot\mathbf{F}_i = -\nabla^2 V_i = 0,$$

we have also

$$\nabla\cdot\mathbf{F} = -\nabla^2 V = 0.$$

Instead of a number of masses m_i, we can consider a continuous distribution of mass inside a volume τ (Figure 8.1). Let ρ be the mass density of the distribution; then the mass in an element $d\tau$ is $\rho\, d\tau$. The gravitational potential at P due to this mass $\rho\, d\tau$ is $-(G\rho/r)\, d\tau$. Then the total gravitational potential at P due to the whole mass distribution is the triple integral over the volume τ:

$$V = -\iiint_{\text{volume }\tau} \frac{G\rho\, d\tau}{r}. \tag{8.3}$$

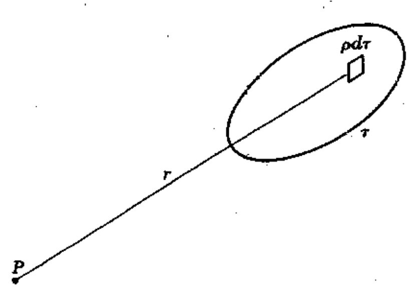


FIGURE 8.1

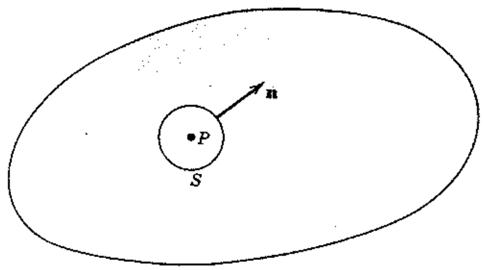


FIGURE 8.2

As before, the contribution to V at P due to each bit of mass satisfies Laplace's equation and therefore V satisfies Laplace's equation. Also the total field $\mathbf{F}$ at P is the vector sum of the fields due to the elements of mass, and as before we have

$$\nabla \cdot \mathbf{F} = -\nabla^2 V = 0.$$

Again note that we are implicitly assuming that none of the mass distribution coincides with P, that is, that $r \neq 0$, which means that point P is not a point of the region τ.

Now let us investigate what happens if P *is* a point of τ. Can we find V from (8.3) and does V satisfy Laplace's equation? Let S be a small sphere of radius a about P; imagine all the mass removed from S (Figure 8.2). Then our previous discussion holds at points inside S since these points are not in the mass distribution. If $\mathbf{F}'$ and V' are the new field and potential (with the matter in S removed), then $\nabla \cdot \mathbf{F}' = -\nabla^2 V' = 0$ at points of S. Now restore the mass to S; let $\mathbf{F}$ and V represent the field and potential due to the whole distribution and let $\mathbf{F}_S$ and V_S represent the field and potential due to just the mass in S. Then

$$\mathbf{F} = \mathbf{F}' + \mathbf{F}_S$$

and at points inside S

$$\nabla \cdot \mathbf{F} = \nabla \cdot \mathbf{F}' + \nabla \cdot \mathbf{F}_S = \nabla \cdot \mathbf{F}_S \tag{8.4}$$

since $\nabla \cdot \mathbf{F}' = 0$ in S.

By the divergence theorem (see Figure 8.2 and Chapter 6, Section 10)

$$\iiint\limits_{\substack{\text{volume}\\ \text{of } S}} \nabla \cdot \mathbf{F}_S \, d\tau = \iint\limits_{\substack{\text{surface}\\ \text{of } S}} \mathbf{F}_S \cdot \mathbf{n} \, d\sigma. \tag{8.5}$$

If we let the radius a of S tend to zero, the density ρ of matter in S tends to its value at P; thus for small a, S contains a total mass M approximately equal to $\frac{4}{3}\pi a^3 \rho$, where ρ is evaluated at P. The gravitational field at the surface of S due to this mass is of magnitude

$$F_S = \frac{GM}{a^2} = G \frac{4}{3} \pi a \rho$$

directed *toward* P. Thus in (8.5), $\mathbf{F}_S \cdot \mathbf{n} = -\frac{4}{3}G\pi a \rho$ because $\mathbf{F}_S$ and $\mathbf{n}$ are antiparallel. Since F_S is constant over the surface S the right-hand side of (8.5) is $\mathbf{F}_S \cdot \mathbf{n}$ times the area of the sphere. The left-hand side is, for small a, approximately the value of $\nabla \cdot \mathbf{F}_S$ at P times the volume of S. Then we have

$$(\nabla \cdot \mathbf{F}_S)(\tfrac{4}{3}\pi a^3) = (-\tfrac{4}{3}G\pi a \rho)(4\pi a^2)$$

or

$$\nabla \cdot \mathbf{F}_S = -4\pi G \rho \quad \text{at} \quad P. \tag{8.6}$$

Since

$$\nabla \cdot \mathbf{F}_S = \nabla \cdot \mathbf{F} = -\nabla \cdot \nabla V = -\nabla^2 V,$$

we have

$$\nabla^2 V = 4\pi G \rho. \tag{8.7}$$

This is Poisson's equation; we see that the gravitational potential in a region containing matter satisfies Poisson's equation as claimed in (1.2). Note that if $\rho = 0$, (8.7) becomes (8.2) as it should.

Next we must consider whether our formula (8.3) for V is valid when P is a point of the mass distribution. The integral appears to diverge at $r = 0$, but this is not really so as we see most easily by using spherical coordinates. Then (8.3) becomes

$$V = -\iiint\limits_{\text{volume } \tau} \frac{G\rho}{r} r^2 \sin\theta \, dr \, d\theta \, d\phi$$

and we see that there is no trouble when $r = 0$. Thus (8.3) is valid in general and gives a solution for (8.7).

Using the notation of (1.2) for Poisson's equation [that is, replacing $4\pi G\rho$ by f and V by u in (8.7) and (8.3)] we can write

$$u = -\frac{1}{4\pi} \iiint \frac{f \, d\tau}{r} \quad \text{is a solution of} \quad \nabla^2 u = f. \tag{8.8}$$

In the more detailed notation needed when we use this solution in a problem, (8.8) becomes (see Figure 8.3):

$$u(x, y, z) = -\frac{1}{4\pi} \iiint \frac{f(x', y', z')}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} \, dx' \, dy' \, dz' \tag{8.9}$$

is a solution of

$$\nabla^2 u(x, y, z) = f(x, y, z).$$

In (8.9) and Figure 8.3, the point (x, y, z) is the point at which we are calculating the potential u; the point (x', y', z') is a point in the mass distribution over which we integrate; r in (8.8) is the distance between these two points and is written out in full in (8.9).

Equations (8.8) or (8.9) actually give a very special solution of Poisson's equation. Recall that it is customary to take the zero point for gravitational (and electrostatic) potential energy at infinity, and this is what we have done. Thus (8.8) or (8.9) gives a

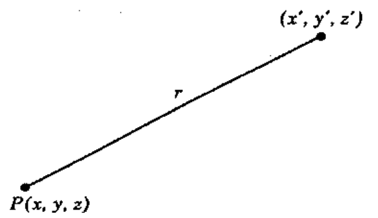


FIGURE 8.3

solution of Poisson's equation which tends to zero at infinity. In another problem this may not be what we want. For example, suppose we have an electrostatic charge distribution near a grounded plane. The electrostatic potential satisfies Poisson's equation, but here we want a solution which is zero on the grounded plane rather than at infinity. To see how we might find such a solution, observe that if u is a solution of Poisson's equation, and w is any solution of Laplace's equation ($\nabla^2 w = 0$), then

(8.10) $$\nabla^2(u + w) = \nabla^2 u + \nabla^2 w = \nabla^2 u = f;$$

thus $u + w$ is a solution of Poisson's equation. Then we can add to the solution (8.9) any solution of Laplace's equation; the combination must be adjusted to fit the given boundary conditions just as we have done in the problems in previous paragraphs.

Example 1. Let us do the following simple problem to illustrate this process. In Figure 8.4, a point charge q at $(0, 0, a)$ is outside a grounded sphere of radius R and center at the origin. Our problem is to find the electrostatic potential V at points outside the sphere. The potential V and the charge density ρ are related by Poisson's equation

(8.11) $$\nabla^2 V = -4\pi\rho \qquad \text{(in Gaussian units).}$$

The potential at (x, y, z) due to a given charge distribution ρ is given by (8.8) or (8.9) with $f = -4\pi\rho$:

(8.12) $$V(x, y, z) = -\frac{1}{4\pi}\iiint \frac{-4\pi\rho(x', y', z')}{\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}}\,dx'\,dy'\,dz'.$$

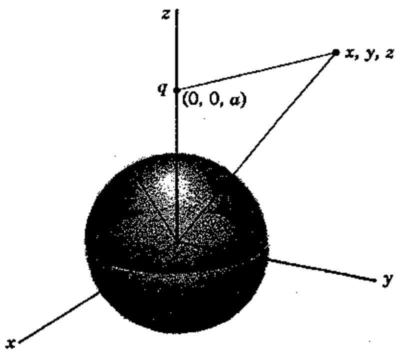


FIGURE 8.4

For a given space-charge distribution, we would next evaluate this integral. For the single point charge q, we have $(x', y', z') = (0, 0, a)$ and we replace $\iiint \rho\,dx'\,dy'\,dz'$ (which is simply the total charge) by q to obtain

(8.13) $$V = \frac{q}{\sqrt{x^2 + y^2 + (z - a)^2}}.$$

[We could, of course, simply have written down (8.13) without using (8.8); (8.13) is just the electrostatic formula corresponding to the gravitational formula (8.1) with which we started.]

Now we want to add to (8.13) a solution of Laplace's equation such that the combination is zero on the given sphere (Figure 8.4). It will be convenient to change to spherical coordinates and to use solutions of Laplace's equation in spherical coordinates. [Note a change in the meaning of r from now on. We have been using r to mean the distance from q at (x', y', z') to (x, y, z); from now on we want to use it to mean the distance from $(0, 0, 0)$ to (x, y, z). See, for example, Figures 8.3 and 8.4.] Writing V_q for V in (8.13) (to distinguish it from our final answer which will be a sum of V_q and a solution of Laplace's equation) and changing to spherical coordinates, we get

(8.14) $$V_q = \frac{q}{\sqrt{r^2 - 2ar\cos\theta + a^2}}.$$

The basic solutions of Laplace's equation in spherical coordinates are (Section 7):

(8.15) $$\begin{Bmatrix} r^l \\ r^{-l-1} \end{Bmatrix} P_l^m(\cos\theta) \begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix}.$$

Since we are interested in the region outside the sphere, we want r solutions which do not become infinite at infinity; thus we use r^{-l-1} and discard the r^l solutions. Because the physical problem is symmetric about the z axis, we look for solutions independent of ϕ; that is, we choose $m = 0$, $\cos m\phi = 1$. Then the basic solutions for our problem are $r^{-l-1}P_l(\cos\theta)$ and we try to find a solution of the form

(8.16) $$V = V_q + \sum_l c_l r^{-l-1} P_l(\cos\theta).$$

We must satisfy the boundary condition $V = 0$ when $r = R$. This gives

(8.17) $$V_{r=R} = \frac{q}{\sqrt{R^2 - 2aR\cos\theta + a^2}} + \sum_l c_l R^{-l-1} P_l(\cos\theta) = 0.$$

Thus we want to expand V_q in a Legendre series. Since V_q is essentially the generating function for Legendre polynomials, this is very easy. Comparing (8.17) and the formulas of Chapter 12, Section 5 [(5.1) and (5.2), or more simply, (5.12) and (5.17)], we find

(8.18) $$\frac{q}{\sqrt{R^2 - 2aR\cos\theta + a^2}} = q\sum_l \frac{R^l P_l(\cos\theta)}{a^{l+1}}.$$

Thus the coefficients c_l in (8.17) are given by

$$c_l R^{-l-1} = -\frac{qR^l}{a^{l+1}} \quad \text{or} \quad c_l = -\frac{qR^{2l+1}}{a^{l+1}}. \tag{8.19}$$

Substituting (8.19) into (8.16), we obtain the final solution for V:

$$V = \frac{q}{\sqrt{r^2 - 2ar\cos\theta + a^2}} - q\sum_l \frac{R^{2l+1}r^{-l-1}P_l(\cos\theta)}{a^{l+1}}. \tag{8.20}$$

Since the second term in (8.20) is of the same general form as (8.18), we can simplify (8.20) by summing the series to get (Problem 2)

$$V = \frac{q}{\sqrt{r^2 - 2ar\cos\theta + a^2}} - \frac{(R/a)q}{\sqrt{r^2 + (R^2/a)^2 - 2r(R^2/a)\cos\theta}}. \tag{8.21}$$

Formula (8.21) has a very interesting physical interpretation. The second term is the potential of a charge $-(R/a)q$ at the point $(0, 0, R^2/a)$; thus we could replace the grounded sphere by this charge and have the same potential for $r > R$. This result can be shown also by elementary analytic geometry and is known as the "method of images." For problems with simple geometry (involving planes, spheres, circular cylinders), it may offer a simpler method of solution than the one we have discussed; however, our purpose was to illustrate the more general method. (Also see Chapter 15, Section 8.)

PROBLEMS, SECTION 8

1. Show that the gravitational potential $V = -Gm/r$ satisfies Laplace's equation, that is, show that $\nabla^2(1/r) = 0$ where $r^2 = x^2 + y^2 + z^2$, $r \neq 0$. (Also see Chapter 15, Section 8.)
2. Using the formulas of Chapter 12, Section 5, sum the series in (8.20) to get (8.21).
3. Do the problem in Example 1 for the case of a charge q inside a grounded sphere to obtain the potential V inside the sphere. Sum the series solution and state the image method of solving this problem.
4. Do the two-dimensional analogue of the problem in Example 1. A "point charge" in a plane means physically a uniform charge along an infinite line perpendicular to the plane; a "circle" at zero potential means an infinitely long circular cylinder perpendicular to the plane. However, since all cross sections of the parallel line and cylinder are the same, the problem is a two-dimensional one. *Hint:* The potential must satisfy Laplace's equation in charge-free regions. What are the solutions of the two-dimensional Laplace equation?
5. Find the method of images for Problem 4.

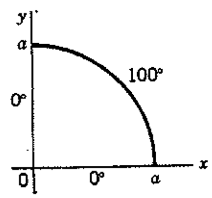
9. MISCELLANEOUS PROBLEMS

1. Find the steady-state temperature distribution in a rectangular plate covering the area $0 < x < 1$, $0 < y < 2$, if $T = 0$ for $x = 0$, $x = 1$, $y = 2$, and $T = 1 - x$ for $y = 0$.
2. Solve Problem 1 if $T = 0$ for $x = 0$, $x = 1$, $y = 0$, and $T = 1 - x$ for $y = 2$. *Hint:* Use sinh ky as the y solution; then $T = 0$ when $y = 0$ ...

3. Solve Problem 1 if the sides $x = 0$ and $x = 1$ are insulated (see Problems 2.14 and 2.15), and $T = 0$ for $y = 2$, $T = 1 - x$ for $y = 0$.
4. Find the steady-state temperature distribution in a plate with the boundary temperatures $T = 30°$ for $x = 0$ and $y = 3$, $T = 20°$ for $y = 0$ and $x = 5$. *Hint:* Subtract 20° from all temperatures and solve the problem; then add 20°. (Also see Problem 2.)
5. A bar of length l is initially at 0°. From $t = 0$ on, the ends are held at 20°. Find $u(x, t)$ for $t > 0$.
6. Do Problem 5 if the $x = 0$ end is insulated and the $x = l$ end held at 20° for $t > 0$. (See Problem 3.9.)
7. Solve Problem 2 if the sides $x = 0$ and $x = 1$ are insulated.
8. A slab of thickness 10 cm has its two faces at 10° and 20°. At $t = 0$, the face temperatures are interchanged. Find $u(x, t)$ for $t > 0$.
9. A string of length l has initial displacement $y_0 = x(l - x)$. Find the displacement as a function of x and t.
10. Solve Problem 5.7 if half the curved surface of the cylinder is held at 100° and the other half at $-100°$ with the ends at 0°.
11. The series in Problem 5.12 can be summed (see Problem 2.6). Show that

$$u = 50 + \frac{100}{\pi}\arctan\frac{2ar\sin\theta}{a^2 - r^2}.$$

12. A plate in the shape of a quarter circle has boundary temperatures as shown. Find the interior steady-state temperature $u(r, \theta)$. (See Problem 5.12.)



13. Sum the series in Problem 12 to get

$$u = \frac{200}{\pi}\arctan\frac{2a^2r^2\sin 2\theta}{a^4 - r^4}.$$

Hint: See Problem 2.6.

14. A long cylinder has been cut into quarter cylinders which are insulated from each other; alternate quarter cylinders are held at potentials $+100$ and -100. Find the electrostatic potential inside the cylinder. *Hints:* Do you see a relation to Problem 12 above? Also see Problem 5.12.
15. Repeat Problems 12 and 13 for a plate in the shape of a circular sector of angle 30° and radius 10 if the boundary temperatures are 0° on the straight sides and 100° on the circular arc. Can you then state and solve a problem like 14?
16. Consider the normal modes of vibration for a square membrane of side π (see Problem 6.3). Sketch the 2,1 and 1,2 modes. Show that the combination $\sin x \sin 2y - \sin 2x \sin y$ of these two modes has a nodal line along $y = x$. Thus find a vibration frequency of a membrane in the shape of a 45° right triangle.

17. Sketch some of the normal modes of vibration for a semicircular drumhead and find the characteristic vibration frequencies as multiples of the fundamental for the corresponding circular drumhead.

18. Repeat Problem 17 for a membrane in the shape of a circular sector of angle 60°.

19. A long conducting cylinder is placed parallel to the z axis in an originally uniform electric field in the negative x direction. The cylinder is held at zero potential. Find the potential in the region outside the cylinder. *Hints:* See Problem 7.13. You want solutions of Laplace's equation in polar coordinates (Problem 5.12).

20. Use Problem 7.16 to find the characteristic vibration frequencies of sound in a spherical cavity.

21. The surface temperature of a sphere of radius 1 is held at $u = \sin^2\theta + \cos^3\theta$. Find the interior temperature $u(r, \theta, \phi)$.

22. Find the interior temperature in a hemisphere if the curved surface is held at $u = \cos\theta$ and the equatorial plane at $u = 1$.

23. Find the steady-state temperature in the region between two spheres $r = 1$ and $r = 2$ if the surface of the outer sphere has its upper half held at 100° and its lower half at −100° and these temperatures are reversed for the inner sphere. *Hint:* See Problem 7.14. Here you will need to find two Legendre series (when $r = 1$ and when $r = 2$) and solve for a_l and b_l.

24. Find the general solution for the steady-state temperature in Figure 2.2 if the boundary temperatures are the constants $T = A$, $T = B$, etc., on the four sides, and the rectangle covers the area $0 < x < a$, $0 < y < b$. *Hints:* You can subtract, say, A from all four temperatures, solve the problem, and then add A back again. Thus a solution with one side at $T = 0$ and the other three at given temperatures solves the general problem. You have previously solved problems (Section 2) with temperatures C and D given. For B, see Problem 2.

25. The Klein-Gordon equation is $\nabla^2 u = (1/v^2)\,\partial^2 u/\partial t^2 + \lambda^2 u$. This equation is of interest in quantum mechanics, but it also has a simpler application. It describes, for example, the vibration of a stretched string which is embedded in an elastic medium. Separate the one-dimensional Klein-Gordon equation and find the characteristic frequencies of such a string.

Answer: $\nu_n = \frac{v}{2}\sqrt{(n/l)^2 + (\lambda/\pi)^2}$.

26. Find the characteristic frequencies of a circular membrane which satisfies the Klein-Gordon equation (Problem 25). *Hint:* Separate the equation in two dimensions in polar coordinates.

27. Do Problem 26 for a rectangular membrane.

14

FUNCTIONS OF A COMPLEX VARIABLE

1. INTRODUCTION

In Chapter 2 we discussed plotting complex numbers $z = x + iy$ in the complex plane (see Figure 1.1) and finding values of the elementary functions of z such as roots, trigonometric functions, logarithms, etc. Now we want to discuss the calculus of functions of z, differentiation, integration, power series, etc. As you know from such topics as differential equations, Fourier series and integrals, mechanics, electricity, etc., it is often very convenient to use complex expressions. The basic facts and theorems about functions of a complex variable not only simplify many calculations but often lead to a better understanding of a problem and consequently to a more efficient method of solution. We are going to state some of the basic definitions and theorems of the subject (omitting the longer proofs), and show some of their uses.

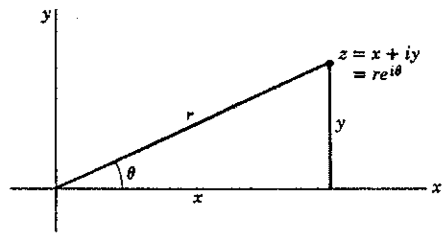


FIGURE 1.1

As we saw in Chapter 2, the value of a function of z for a given z is a complex number. Consider a simple function of z, namely $f(z) = z^2$. We may write

$$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy = u(x, y) + iv(x, y),$$

where $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. In Chapter 2, we observed that a complex number $z = x + iy$ is equivalent to a pair of real numbers x, y. Here we may note that