(1) Plug $v_0(\theta)$ and $v_1(\theta)$ into equation:

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d1}{d\theta}\right) = 0 = -\lambda_0 v_0 \Rightarrow \lambda_0 = 0$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\cos\theta}{d\theta}\right) = -\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin^2\theta\right)$$

$$= -\frac{2\sin\theta\cos\theta}{\sin\theta} = -2\cos\theta = -\lambda_1 v_1$$

$$\Rightarrow \lambda_1 = 2$$

(2) • The boundary conditions are independent of ϕ

→ we expect the solution to be independent of ϕ as well.

• The boundary conditions can be written as

$\sum_i' \alpha_n v_n(\theta)$ with $v_n(\theta) =$ some eigenfunctions of the θ-operator

⇒ we expect the solution to have the same property

$$\rightarrow T(r,\theta,\phi) = \tilde{T}(r,\theta) \quad \leftarrow \text{no } \phi \text{ dependence}$$

$$= f_0(r)v_0(\theta) + f_1(r)v_1(\theta)$$

$$= f_0(r) + f_1(r)\cos\theta.$$

Plugging this into the equation ⇒

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df_0}{dr}\right) + \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df_1}{dr}\right)\cos\theta + \frac{1}{r^2}\frac{f_1(r)}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\cos\theta}{d\theta}\right) = 0$$

$$\Leftrightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df_0}{dr}\right) + \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df_1}{dr}\right)\cos\theta - \frac{2}{r^2}f_1(r)\cos\theta = 0$$

which implies

$$\begin{cases} \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df_0}{dr}\right) = 0 \\ \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{df_1}{dr}\right) - \frac{2}{r^2}f_1 = 0 \end{cases}$$

$$r^2 \frac{df_0}{dr} = \text{constant} = A$$

$$\Rightarrow \frac{df_0}{dr} = \frac{A}{r^2} \Rightarrow f_0 = -\frac{A}{r} + B$$

Solving for f_1: try $f_1 = r^\alpha$: $\frac{d^2 f_1}{dr^2} + \frac{2}{r}\frac{df_1}{dr} - \frac{2}{r^2} f_1 = 0$

then $\alpha(\alpha-1) + 2\alpha - 2 = 0$

$$\Rightarrow \alpha(\alpha-1) + 2(\alpha-1) = 0$$

$$\Leftrightarrow (\alpha+2)(\alpha-1) = 0 \Rightarrow \begin{cases} \alpha = 1 \\ \alpha = -2 \end{cases}$$

So $$f_1 = Cr + \frac{D}{r^2}$$

So finally $T(r,\theta,\phi) = -\frac{A}{r} + B + \left(Cr + \frac{D}{r^2}\right)\cos\theta$.

③ Fitting to boundary conditions:

$T(a,\theta,\phi) = T_0 \Leftrightarrow$

$$-\frac{A}{a} + B + \left(Ca + \frac{D}{a^2}\right)\cos\theta = T_0$$

$$\Rightarrow \begin{cases} -\frac{A}{a} + B = T_0 & (*) \\ Ca + \frac{D}{a^2} = 0 & (**) \end{cases}$$

$T(b,\theta,\phi) = T_1 + T_2\cos\theta \Leftrightarrow$

$$-\frac{A}{b} + B + \left(Cb + \frac{D}{b^2}\right)\cos\theta = T_1 + T_2\cos\theta$$

$$\Rightarrow \begin{cases} -\frac{A}{b} + B = T_1 & (\dagger) \\ Cb + \frac{D}{b^2} = T_2 & (\dagger\dagger) \end{cases}$$

From $(**) \Rightarrow C = -\frac{D}{a^3}$

$(\dagger\dagger) \Rightarrow -\frac{Db}{a^3} + \frac{D}{b^2} = T_2 \Rightarrow D = \dfrac{T_2}{\frac{1}{b^2} - \frac{b}{a^3}}$

From $(*) - (\dagger)$: $-\frac{A}{a} + \frac{A}{b} = T_0 - T_1$

$$\Rightarrow A = \frac{T_0 - T_1}{\cdots} = \frac{ab(T_0 - T_1)}{\cdots}$$

and finally $B = T_0 + \frac{A}{a} = T_0 + \frac{b(T_0-T_1)}{a-b}$

$$\Rightarrow \quad T(r,\theta,\phi) = -\frac{ab(T_0-T_1)}{a-b}\frac{1}{r} + T_0 + \frac{b(T_0-T_1)}{a-b}$$
$$+ \left(-\frac{1}{a^3}r + \frac{1}{r^2}\right)\left(\frac{T_2}{\frac{1}{b^2}-\frac{b}{a^3}}\right)\cos\theta$$

Note: simplify with $D = \frac{T_2 b^2 a^3}{a^3-b^3}$

$$C = -\frac{D}{a^3} = \frac{T_2 b^2}{b^3-a^3}$$

$$\Rightarrow \quad T(r,\theta,\phi) = T_0 + \frac{b(T_1-T_0)}{b-a}\left(1-\frac{a}{r}\right)$$
$$+ \left(\frac{1}{r^2}-\frac{r}{a^3}\right)\frac{T_2 b^2 a^3}{a^3-b^3}\cos\theta$$

$$= T_0 + \frac{b(T_1-T_0)}{b-a}\left(1-\frac{a}{r}\right)$$
$$+ \left(\frac{a^2}{r^2}-\frac{r}{a}\right)\frac{T_2 b^2 a}{a^3-b^3}\cos\theta$$

<u>Check</u>: at $r=a$: $T = T_0$ ✓

at $r=b$ $T = T_0 + \frac{b(T_1-T_0)}{b-a}\left(1-\frac{a}{b}\right)$
$$+ \left(\frac{a^2}{b^2}-\frac{b}{a}\right)\frac{T_2 b^2 a}{a^3-b^3}\cos\theta$$

$$= \cancel{T_0} + T_1 - \cancel{T_0} + T_2\cos\theta$$

$$= T_1 + T_2\cos\theta.$$ ✓