FINAL 2007

Problem 1 : see Final 2006.

Problem 2 :

$$\begin{cases} u_t - k u_{xx} = t \sin\left(\frac{3\pi x}{L}\right) & x \in [0, L] \\ u(x,0) = 0 & t \geq 0. \\ u(0,t) = u(L,t) = 0. \end{cases}$$

Consider the solutions of $u_{xx} = -\lambda u$ satisfying the bcs:

$$\rightarrow \quad v_n(x) = \sin\left(\frac{n\pi x}{L}\right) \quad \lambda_n = \frac{n^2\pi^2}{L^2}$$

so let's write $\begin{cases} t \sin\left(\frac{3\pi x}{L}\right) = \sum b_n(t) v_n(x) \\ u(x,t) = \sum a_n(t) v_n(x) \end{cases}$

$\Rightarrow \quad b_3(t) = t$, all others are 0

and $\quad \dot{a}_n + k \frac{n^2\pi^2}{L^2} a_n = b_n(t) \quad \forall n.$

for $n \neq 3$, $\quad a_n(t) = a_n(0) e^{-\frac{n^2\pi^2}{L^2} kt}$

for $n = 3$, $\quad a_3(t) = K e^{-\frac{9\pi^2}{L^2} kt}$ + part. int.

for the particular integral, try $\alpha t + \beta$

$$\Rightarrow \quad \alpha + \frac{9k\pi^2}{L^2}(\alpha t + \beta) = t$$

so $\quad \frac{9\alpha k\pi^2}{L^2} = 1 \qquad \Rightarrow \qquad \alpha = \frac{L^2}{9k\pi^2}$

$\quad \alpha + \frac{9k\pi^2}{L^2}\beta = 0 \qquad \beta = -\frac{L^4}{81k^2\pi^4}$

So eventually

$$a_3(t) = K e^{-\frac{9\pi^2}{L^2} kt} + \frac{L^2 t}{9k\pi^2} - \frac{L^4}{81k^2\pi^4}$$

So

$$u(x,t) = \sum' a_n(t) V_n(x) \text{ with } a_n(t) \text{ defined above.}$$

Fitting to initial conditions requires

$$\sum' a_n(0) V_n(x) = 0$$

$$\Rightarrow \quad a_n(0) = 0 \quad \text{for all } n$$

$$\Rightarrow \quad K = \frac{L^4}{81k^2\pi^4} \quad \text{so}$$

$$u(x,t) = \left\{ \frac{L^4}{81k^2\pi^4} \left(e^{-\frac{9\pi^2 kt}{L^2}} - 1 \right) + \frac{L^2 t}{9k\pi^2} \right\} \sin\left(\frac{3\pi x}{L}\right).$$

Problem 3

- General solution of the pb: see textbook (handout) (eq. 6.6)

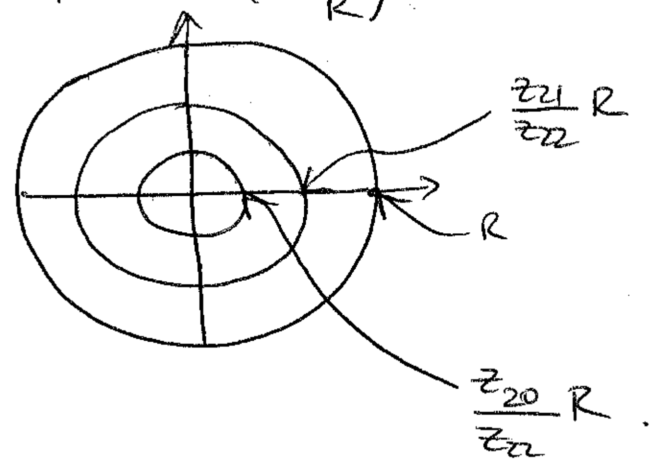
$$u(r,\theta,t) = \sum_{n,m}' J_n(k_{nm} r)\left(a_{nm}\cos(n\theta) + b_{nm}\sin(n\theta)\right) \cdot \left(\alpha_{nm}\cos(k_{nm}ct) + \beta_{nm}\sin(k_{nm}ct)\right)$$

where k_{nm} are $\frac{z_{nm}}{R}$.

- $\Rightarrow \quad \nu_{nm} = \frac{k_{nm}c}{2\pi} = \frac{z_{nm}c}{2\pi R} = \frac{z_{nm}}{\tau}$

where $\tau = \frac{2\pi R}{c}$

Nodal lines of $J_2\left(z_{22}\frac{r}{R}\right)\sin 2\theta$



Problem 4

$$\begin{cases} \nabla^2 T = H(x,y). \\ T = 0 \text{ on contour } (x=1, y=1, x=0, y=0) \end{cases}$$

$\rightarrow \quad T_{xx} + T_{yy} = H(x,y).$

1. Let's solve the SL problem $u_{xx} = -\lambda u$ with
$u(x=0) = u(x=1) = 0$

$\rightarrow \quad u_n(x) = \sin(n\pi x)$
$\lambda_n = n^2\pi^2$

then write $\quad T(x,y) = \sum_1 a_n(y)\, u_n(x)$
$H(x,y) = \sum_1 b_n(y)\, u_n(x)$

with $\quad b_n(y) = 2\int_0^1 H(x,y)\sin(n\pi x)\,dx.$

we then have
$$\ddot{a}_n - n^2\pi^2 a_n = b_n(y).$$

So $\quad a_n(y) = \alpha_n e^{n\pi y} + \beta_n e^{-n\pi y} + \text{P.I.}$

To find the P. I. note that

$$b_n(y) = \sum_m \gamma_{mn} \sin(m\pi y)$$

where $\gamma_{mn} = 2\int_0^1 b_n(y) \sin(m\pi y)\, dy$

$$= 4 \int_0^1\!\!\int_0^1 H(x,y) \sin(m\pi y) \sin(n\pi x)\, dy\, dx$$

Trying a PI of the form $\sum_m k_m \sin(m\pi y)$ we get

a_n $\quad PI = \sum_m - \dfrac{\gamma_{mn}}{m^2+n^2} \sin(m\pi y).$

so that $\quad a_n(y) = \alpha_n e^{n\pi y} + \beta_n e^{-n\pi y} - \sum_m' \dfrac{\gamma_{mn}}{m^2+n^2} \sin(m\pi y)$

$a_n(0) = 0 \quad \Rightarrow \quad \alpha_n + \beta_n = 0. \quad \alpha_n = -\beta_n$

$a_n(1) = 0 \quad \Rightarrow \quad \alpha_n e^{n\pi} + \beta_n e^{-n\pi} = 0 \quad \Rightarrow \quad \alpha_n = \beta_n = 0.$

So $\quad a_n(y) = -\sum_m' \dfrac{\gamma_{mn}}{m^2+n^2} \sin(m\pi y).$

and finally

$$T(x,y) = -\sum_{n,m}' \frac{\gamma_{mn}}{m^2+n^2} \sin(n\pi x) \sin(m\pi y).$$

which is of the form required provided

$$a_{mn} = -\frac{\gamma_{mn}}{m^2+n^2} = -\frac{4}{m^2+n^2} \int_0^1\!\!\int_0^1 H(x,y) \sin(m\pi y) \cdot \sin(n\pi x)\, dx\, dy$$

2. This is of the form $T(x,y) = \int_0^1\!\!\int_0^1 H(x',y') G(x,x';y,y')\, dx'\, dy'$

provided

$$G(x,x';y,y') = \sum_{m,n}' \frac{4}{m^2+n^2} \sin(n\pi x) \sin(n\pi x') \sin(m\pi y) \sin(m\pi y')$$

→ The Greens function.

If $H(x,y) = \sin^2(\pi x)\sin^2(\pi y)$ then

$$T(x,y) = \int_0^1\!\!\int_0^1 \sum_{m,n} -\frac{4}{m^2+n^2} \sin(n\pi x)\sin(m\pi y)\sin^2(\pi x')\sin^2(\pi y')\sin(n\pi x') \cdot \sin(m\pi y')\,dx'\,dy'$$

Since $\displaystyle\int_0^1 \sin^2(\pi u)\sin(n\pi u)\,du = \frac{2(-1+\cos(n\pi))}{n\pi(n^2-4)}$ if $n \neq 2$ and 0 otherwise

then

$$T(x,y) = \sum_{\substack{m,n \\ \text{both} \neq 2}} -\frac{4}{m^2+n^2} \sin(n\pi x)\sin(m\pi y)\, \frac{2(-1+\cos(n\pi))}{n\pi(n^2-4)}\, \frac{2(-1+\cos(m\pi))}{m\pi(m^2-4)}.$$