Final

Instructions:

- All problems are worth the same number of points. Attempt as many as you can.
- Calculators are allowed but shouldn't be necessary.
- MAKE SURE YOUR NAME IS ON THE FIRST PAGE OF YOUR ANSWERS, AND YOUR INITIALS ON ALL OTHER PAGES
- The instructor reserves the right not to grade any illegible answers.
- Note: solving 3 out of the 4 problems completely and correctly will garantee an A^+.
- RELAX, AND GOOD LUCK!

Problem 1: This problem is aimed at finding the equilibrium temperature profile within a spherical cavity. We consider a region bounded by two concentric spheres of radius a and b respectively (with $a < b$). We use a spherical coordinate system (r, θ, ϕ). The temperature on the inner sphere is held at a constant value T_0

$$T(a, \theta, \phi) = T_0 \tag{1}$$

while the temperature on the outer sphere is

$$T(b, \theta, \phi) = T_1 + T_2 \cos(\theta) \tag{2}$$

The steady-state temperature profile is obtained by solving

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2} = 0 \tag{3}$$

subject to the boundary conditions required.

- Verify that the functions $v_0(\theta) = 1$ and $v_1(\theta) = \cos\theta$ are eigen-solutions of the equation

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta \frac{\mathrm{d}v_i}{\mathrm{d}\theta}\right) = -\lambda_i v_i \tag{4}$$

  and find the corresponding eigenvalues λ_0 and λ_1.

- Fully justify that the solution to the problem can be written as

$$T(r, \theta, \phi) = f_0(r) + f_1(r)\cos\theta \tag{5}$$

  where $f_0(r)$ and $f_1(r)$ satisfy (for $i = 0$ or $i = 1$)

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}f_i}{\mathrm{d}r}\right) - \frac{\lambda_i}{r^2} f_i = 0 \tag{6}$$

- Solve the equations for $f_0(r)$ and $f_1(r)$ and identify the arbitrary constants that will need to be determined from the boundary conditions. (Hint: seek solutions in r^α).

- By applying the boundary conditions, find $T(r, \theta, \phi)$.

Problem 2: Solve the problem

$$\begin{aligned}
&u_t - k u_{xx} = t \sin\left(\frac{3\pi x}{L}\right), \qquad 0 < x < L, t > 0 \\
&u(x, 0) = 0, \qquad 0 < x < L \\
&u(0, t) = u(L, t) = 0
\end{aligned}$$

Problem 3: We consider a circular drum of radius R oscillating according to the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right] \tag{7}$$

where c is a constant, and (r, θ) are the standard plane-polar coordinates. The boundary conditions near the outer rim of the drum are such that $u(R, t) = 0$, and regularity conditions are required at the origin ($r = 0$). We seek to find all the possible eigenmodes and eigenfrequencies of the drum.

- Find the general solution of the problem (irrespective of initial conditions).
- Let $\{z_{nm}\}_{m=0..\infty}$ be the ensemble of all the roots of $J_n(x)$ (i.e. $J_n(z_{nm}) = 0$). Given that the frequency of oscillation ν of the periodic function $\sin(\omega t)$ (or $\cos(\omega t)$) is $\nu = 2\pi/\omega$, what are all of the possible eigen-frequencies of the drum?
- The nodal lines of an eigenmode are lines where the eigenmode is zero. Sketch the nodal lines of $\sin(2\theta) J_2 \left(z_{22} \frac{r}{R} \right)$.

Note: you may need the following information on Bessel functions:

- The Bessel equation
$$x^2 f_{xx} + x f_x + (x^2 - n^2) f = 0 \tag{8}$$
has the general solution $f(x) = a J_n(x) + b Y_n(x)$ where the functions $J_n(x)$ and $Y_n(x)$ are Bessel functions of the first and second kind respectively. Note that $J_n(x)$ is regular at $x = 0$ while $Y_n(x)$ is singular.
- Here is an annotated plot of $J_2(x)$:

Problem 4: We consider a square metallic plate constantly heated from its surface and cooling at the sides. The length of each side is unity. The equation for the steady-state temperature profile of the plate is given by

$$\nabla^2 T = H(x,y) \tag{9}$$

where $H(x,y)$ is the spatially varying heating term, and ∇^2 is the two-dimensional Laplacian in Cartesian coordinates. The boundary conditions are simply $T(0,y) = T(1,y) = T(x,0) = T(x,1) = 0$, and you may assume that $H(0,y) = H(1,y) = H(x,0) = H(x,1) = 0$ as well.

- Show that the solution to the problem can be written as

$$T(x,y) = \sum_n \sum_m a_{mn} \sin(m\pi x)\sin(n\pi y) \tag{10}$$

  where you need to express the coefficients a_{mn} in terms of the heating function $H(x,y)$.

- If we re-wrote the solution as

$$T(x,y) = \int_{x'=0}^{x'=1} \int_{y'=0}^{y'=1} G(x,y;x',y')H(x',y')\mathrm{d}x'\mathrm{d}y' \tag{11}$$

  what is the name of the function G and what is its exact expression?

- Solve the problem when $H(x,y) = \sin^2(\pi x)\sin^2(\pi y)$.