Take Home Midterm

You need to justify all your answers. Answers without justifications will be counted as wrong. You **may not** ask for help from any other living being aside from the instructor. If you need help from the instructor, make sure to ask **before** Friday, 3pm. Help will be limited to clarifications on the questions if needed.

Problem 1: First order PDEs

(1) Solve the following problem

$$u_t + uu_x = 0$$
$$u(x, 0) = \frac{|x|}{x}$$

using the method learned in class to find the solution in the expansion shock.

We now look at an alternative view of constructing the expansion shock solution.

(2) Consider the line formed by the combination of the $t = -x$ (for $x < 0$) line and the $t = x$ (for $x > 0$) line as the new initial condition curve Γ for a new problem. Write down the parametric representation of Γ as $\{x_0(s),\, t_0(s)\}$. Hint: using absolute values helps.

(3) Express the solution for $u(x, t)$ of question (1) on Γ as the function $u_0(s) = u(x_0(s), t_0(s))$, and solve the PDE again with this new initial condition. Make sure to compare your answers!

Problem 2: The Yule Process

Let's consider the following model for a population of bacteria growing in a petri dish. Let $X(t)$ be the number of bacteria present in the dish at time t, and let $P_n(t)$ be the probability that $X(t) = n$. We are going to make the following assumptions:

- Bacteria do not die.
- A single bacteria gives birth to another bacteria (by cellular division) with probability $\lambda h + o(h)$ during the time interval $(t, t + h)$. The probability that the same bacteria (or its offspring) gives birth to yet another one during the same interval of time is $o(h)$.
- Bacteria given birth independently of one another.
- At time $t = 0$, there are n_0 bacteria in the dish. We know that for sure!

(a) Show that the probabilities $P_n(t)$ satisfy the coupled system of ODEs:

$$P_n'(t) = 0 \text{ for } n < n_0$$
$$P_{n_0}'(t) = -\lambda n_0 P_{n_0}(t)$$
$$P_n'(t) = -\lambda n P_n(t) + \lambda(n-1)P_{n-1}(t) \text{ for } n > n_0$$

(b) Show that the probability generating function $G(t, s) = \sum_n P_n(t)s^n$ satisfies the initial value problem

$$\begin{aligned} \frac{\partial G}{\partial t} &= -s\lambda(1-s)\frac{\partial G}{\partial s} \\ G(0, s) &= s^{n_0} \end{aligned}$$

(c) Solve this problem and show that

$$G(s, t) = \left[\frac{se^{-\lambda t}}{1 - s(1 - e^{-\lambda t})}\right]^{n_0}$$

(d) Deduce that the expectation value of the population size is $n_0 e^{\lambda t}$, and comment on whether this is the result you expected.

Problem 3: Canonical forms

Textbook p. 74, problem 3.8

Problem 4: Damped vibrating string

Consider the problem of a damped vibrating string, which is governed by the equation

$$\frac{\partial^2 h}{\partial t^2} = c^2 \frac{\partial^2 h}{\partial x^2} - b\frac{\partial h}{\partial t}$$

(a) Find the general solution to this PDE given the following boundary and initial conditions:

- $h(0, t) = h(L, t) = 0$
- $h(x, 0) = f(x)$
- $h_t(x, 0) = g(x)$

(b) Apply it to the following case:

$$\begin{aligned} f(x) &= 0 \\ g(x) &= \frac{2v_0 x}{L} \text{ if } x \in [0, L/2] \\ g(x) &= 2v_0 - \frac{2v_0 x}{L} \text{ if } x \in [L/2, L] \end{aligned}$$